

# Mathematical Reviews

*Edited by*

**W. Feller**

**M. H. Stone**

**H. Whitney**

**R. P. Boas, Jr., Executive Editor**

Vol. 10, No. 1

January, 1949

pp. 1-92

This issue appears before the publication of Vol. 9, No. 11 (Index issue).

## TABLE OF CONTENTS

<b>Foundations</b> . . . . .	1	Functional analysis, ergodic theory . . . . .	46
<b>Algebra</b> . . . . .	3	Mathematical statistics . . . . .	50
Abstract algebra . . . . .	4	Mathematical economics . . . . .	52
<b>Theory of groups</b> . . . . .	8	<b>Topology</b> . . . . .	53
<b>Number theory</b> . . . . .	12	<b>Geometry</b> . . . . .	57
<b>Analysis</b> . . . . .	20	Convex domains, extremal problems . . . . .	59
Calculus . . . . .	22	Algebraic geometry . . . . .	60
Theory of sets, theory of functions of real variables . . . . .	22	Differential geometry . . . . .	62
Theory of functions of complex variables . . . . .	24	<b>Numerical and graphical methods</b> . . . . .	67
Theory of series . . . . .	31	<b>Mechanics</b> . . . . .	71
Fourier series and generalisations, integral transforms . . . . .	33	Hydrodynamics, aerodynamics, acoustics . . . . .	72
Polynomials, polynomial approximations . . . . .	37	Elasticity, plasticity . . . . .	80
Special functions . . . . .	38	<b>Mathematical physics</b> . . . . .	88
Harmonic functions, potential theory . . . . .	39	Optics, electromagnetic theory . . . . .	88
Differential equations . . . . .	40	Quantum mechanics . . . . .	90
Integral equations . . . . .	45	Thermodynamics, statistical mechanics . . . . .	91

## AUTHOR INDEX

Abele, M.	90	Bing, R. H.	55	Choquet, G.	53	Egervárt, S.	86
Abele, F.	4	Blaschke, W.	58	Chowla, S. See Bachbach, R. P.	20	Ehrenmann, C.	56
Ahiezer, N. I.	33	Blau, R. E. See Weibel, E. E.	58	Christoff, C.	20	Eilenberg, S. MacLane, S.	5
Ahlfors, L. V.	28	Bilas, C. I. See Cochran, W. G.	58	Cicala, P.	82	Errera, A.	53
Alden, H. L.	76	Boas, R. P. Jr.	58	Ciricuian, J. E.	90	Erdélyi, H.	79
Alder, H. L.	16	Chandrasekharan, K.	21	Clifford, A. H.	12	Ervard, J. C.	78
Aleksandrov, P. S.	53	Bockstein, M.	56	Climescu, A. C.	25	Fabricius-Bjerre, F.	59
Alexiewicz, A.	31	Bodewig, E.	69	Cochran, W. G. Bilas, C. I.	50	Fal'ković, S. V. See	
Amante, S.	4, 15	Boggio, T.	22	Colombo, S. See Kahan, T.	53	Polubarinova-Kochina, P. V.	
Ambarcumyan, S. A.	87	Boldrini, M.	50	Combe, J.	28	Fano, G.	61
Amerio, L.	43	Bomplani, E.	64	Conforto, F.	29	Fazan, O. H.	73
Ammann, A.	17, 18	Bonfiglioli, G.	87	Cooper, J. L. B.	35	Federer, W. T. See	
Anfert'eva, E. A.	17	Bopp, F.	92	Cooper, J. L. B.	70	Kempthorne, O.	
Aprile, G.	36, 70	Bordoni, P. G.	39, 80	Cope, W. F. Hartree, D. R.	74	Federhofer, K.	23
Arens, R. F. Kaplansky, I.	7	Borel, A. de Siebenthal, J.	12	van der Corput, J. G.	69	Feinberg, S. M.	24
Arplarian, N.	24, 25	Borgh, D. C.	91	Correnti, S.	3	Fejér, Tóth, L.	60
Astolfo, E.	43	Borsig, K.	54, 60	Cranks, J.	28	Feldheim, E.	36, 37
Atkinson, F. V.	32	Bouligand, G.	73	Conzett, F.	29	Ferrari, C.	78
Ayer, M. C. Radó, T.	24	Braconnier, J.	11	Cooper, J. L. B.	35	Fichera, G.	41
Beckes, F.	63	Brauer, A.	14	Cooper, J. L. B.	70	Finney, D. J.	52
Bagana, N.	28	de Broglie, L.	91	Cope, W. F. Hartree, D. R.	74	Fiamm, L.	89
Baleiro, J. A.	91	Brown, B. McCoy, N. H.	6	van der Corput, J. G.	69	Flint, H. T. Symonds, N.	91
Bambah, R. P. Chowla, S.	14	Browne, S. H. Friedman, L.	78	Fog, D.	59	Fok, V. A.	89
Banach, S.	23	Hodes, I.	78	Fréchet, M.	50	Freeman, H. A. Friedman, M.	
Banerjee, D. P.	28	de Bruijn, N. G.	16, 23	Decuyper, M.	57	Mosteller, F. Wallis, W. A.	52
Barbasin, E. A.	49	Bullen, K. E.	88	Del Poli, S.	57	Freilich, G.	40
Barber, N. F. Ursell, F.	40	Burgers, J. M.	73	Delange, H.	32	Friede, G. Minzner, H.	50
Barbilian, D.	7	Caloi, P.	88	Delarue, S.	9	Friedman, L. See Browne, S. H.	
Bartlett, A. C.	22	Carlson, F.	27	De Simoni, F.	90	Friedman, M. See	
Bartolo, M.	57	Carrasco, L. E.	24	Deuring, M.	5	Freeman, H. A.	
Battorf, S. B. See Libove, C.		Carrier, G. F.	83	Devaux, P.	3	Friedrichs, K. O.	41
Bates, G. E.	12	Cărstoia, I.	72	Dieudonné, J.	57	Fromm, H.	73
Batachelet, E.	72	Cassels, J. W. S.	19	Dieudonné, C. E.	5	Fuchs, L.	6
Beaumont, R. A.	10	Castoldi, L.	80	Ditkin, V. A.	40	Fuchs, W. H. J.	21
Beckenbach, E. F.	62	Cattaneo, C.	80	Doe, S.	27	Gaeta, F.	61
Bellman, R.	43	Cattaneo, P.	22	Drach, J.	61	Gagiev, B. M.	37
Belluzzi, O.	85, 86	Čebotarev, N. G.	20	Dritgau, M.	90	Gage, W. H.	15
Bergman, S.	30	Cernikov, S. N.	10	Dryden, H. L.	74	Galin, L. A.	84
Bergman, S. Schiffer, M.	42	Chakrabarti, M. C.	50, 70	Dubisch, R. Perlis, S.	6	Galonen, L. M.	41
Bernaya, F.	3	Chakrabarti, S.	3	Dyson, F. J.	19, 55	Garrick, I. E.	77
Bernstein, S. N.	29	Chalk, J. H. H.	18			Gartstein, B. N.	51
Beškin, L.	87	Chandrasekharan, K. See				Gavrilov, M. A.	90
Beth, E. W.	3	Boss, R. P. Jr.				Gerashimov, A. N.	80
Bhatia, A. B. Krishnan, K. S.	35	Chern, Shih-chen.	65			Gerjuoy, E.	80
Bicadze, A. V.	24	Chern, Shih-chen - Wang.				Geromimus, J. L.	26, 37
Biezeno, C. B.	80	Hsién-chung.	65				
Bijlsma, P. P.	87	Ching, K. S. See Lin, T. H.					

(Continued on cover 4)

## ONE-SIDE EDITION OF MATHEMATICAL REVIEWS

An edition of MATHEMATICAL REVIEWS printed on only one side of the paper is available to persons interested in making card files of reviews or wishing to add remarks to reviews in the future. This special edition may be obtained

for an additional payment of \$1.00. A regular current subscription can be changed to a one-side subscription by paying the additional \$1.00. This edition is folded but not stitched.

### MATHEMATICAL REVIEWS

*Published monthly, except August, by*

**THE AMERICAN MATHEMATICAL SOCIETY, Prince and Lemon Streets, Lancaster, Pennsylvania**

*Sponsored by*

THE AMERICAN MATHEMATICAL SOCIETY  
 THE MATHEMATICAL ASSOCIATION OF AMERICA  
 THE INSTITUTE OF MATHEMATICAL STATISTICS  
 THE EDINBURGH MATHEMATICAL SOCIETY  
 L'INTERMÉDIAIRE DES RECHERCHES MATHÉMATIQUES  
 MATEMATISK FORENING I KØBENHAVN  
 HET WISKUNDIG GENOOTSCHAP TE AMSTERDAM  
 THE LONDON MATHEMATICAL SOCIETY  
 POLISH MATHEMATICAL SOCIETY  
 UNIÓN MATEMÁTICA ARGENTINA  
 INDIAN MATHEMATICAL SOCIETY

*Editorial Office*

MATHEMATICAL REVIEWS, Brown University, Providence 12, R. I.

**Subscriptions:** Price \$13 per year (\$6.50 per year to members of sponsoring societies). Checks should be made payable to MATHEMATICAL REVIEWS. Subscriptions should be addressed to MATHEMATICAL REVIEWS, Lancaster, Pennsylvania, or Brown University, Providence 12, Rhode Island.

This publication was made possible in part by funds granted by the Carnegie Corporation of New York, the Rockefeller Foundation, and the American Philosophical Society held at Philadelphia for Promoting Useful Knowledge. Its preparation is also supported currently under a contract with the Office of Naval Research, Department of the Navy, U.S.A. These organizations are not, however, the authors, owners, publishers, or proprietors of this publication, and are not to be understood as approving by virtue of their grants any of the statements made or views expressed therein.

Entered as second-class matter February 3, 1940 at the post office at Lancaster, Pennsylvania, under the act of March 3, 1879. Accepted for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in paragraph 4, section 538, P. L. and R. authorized November 9, 1940.



\*Lor  
i+  
He  
MK  
interpret  
"erlau  
ness,  
part  
axiom  
disjunc  
but th  
tion  
and  
being  
elem  
duced  
the th  
disjunc  
use o

The  
of set  
numb  
etry  
distan  
 $\vec{pq}$ , tr  
a line  
with

Turin  
bol  
Ver  
verein  
schem  
brauc  
bereic  
werde  
1," "  
tionen  
(die a  
den  
» "in  
sagef  
Zur  
aufste  
ist a  
Präm  
Stellt  
Kard  
to for  
to be  
keitsa  
hinz  
Es  
traditi

# Mathematical Reviews

Vol. 10, No. 1

JANUARY, 1949

Pages 1-92

## FOUNDATIONS

\*Lorenzen, Paul. *Einführung in die Logik*. Bonn, 1948. i+34 pp.

Here logic appears as the partly formalized metatheory *MK* of a calculus *K*. However, *K* is given together with an interpretation and as the definitions of such concepts as "erlaubt," "verträglich" are worded with insufficient exactness, it is difficult to see how far this interpretation plays a part in the inferences. For example, *K* consists of the Peano axioms for elementary arithmetic, the signs for equality, disjunction, conjunction and the quantifiers belonging to *K*, but the implication sign to *MK*, so that the axiom of induction must be formulated in *MMK*. The signs of *K*, *MK* and *MMK* are combined freely, every formula evidently being considered as its own name. On this basis a part of elementary arithmetic is developed, a negation sign is introduced in *K* by means of axioms and a proof is sketched of the theorem: any theorem in *K*, which contains no negation, disjunction or existential quantifier, can be deduced without use of the principle of excluded middle.

The second chapter contains the foundations of the theory of sets of natural numbers; the third, the theory of real numbers. In chapter 4 a system of axioms for plane geometry is given in terms of the undefined concepts: point *p*, distance  $\|pq\|$ , straight line *p*  $\vee$  *q*, reflection in a line, vector  $\vec{pq}$ , translation along a vector, orientation of a vector along a line. The axioms are accounted for by mental experiments with solid bodies.

A. Heyting (Amsterdam).

Turing, A. M. Practical forms of type theory. *J. Symbolic Logic* 13, 80-94 (1948).

Verf. versucht die (unverzweigte) Typenlogik so zu vereinfachen, dass sie für die traditionelle, von der logistischen Formalisierung noch nicht ergriffene Mathematik brauchbar wird. Er geht von einem endlichen Individuenbereich aus, zu dem dann die Bereiche der "Funktionen," dargestellt durch Tabellen, schrittweise hinzugenommen werden. Die Elemente dieser Bereiche heißen "Vom Typ 1," "Vom Typ 2," ... Die "Aussagen" werden als Funktionen mit den Wahrheitswerten *T* (True) und *F* (False) (die zu den Individuen gerechnet werden) eingeführt. Werden dann die logischen Operationen (z.B.  $\sim$  "nicht,"  $\supset$  "impliziert" und  $(x, r)$  "für alle *x* vom Typ *r*") als Aussagefunktionen eingeführt, so ist jede Aussage entscheidbar. Zur Erleichterung dieser Entscheidung kann man Regeln aufstellen von der Form: "Wenn  $P_1, \dots, P_n$  wahr sind, so ist auch *P* wahr," z.B.  $P, P \supset q \rightarrow q$ . Die Regeln ohne Prämissen ( $n=0$ ) sind die Tautologien, z.B.  $\sim P \supset \sim P$ . Stellt man die Regeln zusammen, die unabhängig von der Kardinalzahl des Individuenbereichs sind, so ist es "easy to forget the finite universe, and to allow the various rules to become reflex action." Schliesslich wird ein Unendlichkeitsaxiom als Ersatz für den endlichen Individuenbereich hinzugenommen.

Es scheint dem Ref. sicher, dass dieses Vorgehen für die traditionelle Mathematik nicht "easy" ist, da sie in ihrer

Domäne, der Arithmetik und Analysis nicht axiomatisch, sondern konstruktiv vorgeht. Obwohl sich die axiomatische Analysis z. Zeit weithin durchgesetzt hat, muss sie ihre Brauchbarkeit noch theoretisch begründen. Die Berechtigung der Resignation des Verf. vor dieser Aufgabe darf angezweifelt werden.

Mit Benutzung eines neuen präzisen Äquivalenzbegriffes lässt sich zeigen, dass das beschriebene "nested-type system" mit der vereinfachten Typenlogik von Church [*J. Symbolic Logic* 5, 56-68 (1940); diese Rev. 1, 321] äquivalent ist. Zum Schluss wird die Möglichkeit untersucht, den Alloperator ohne Typenbeschränkung einzuführen. Lässt sich (grob gesprochen) für einen Typ  $r_0$  beweisen, dass für alle Typen  $r > r_0$  die Aussagen  $(x, r)P$  und  $(x, r_0)P$  äquivalent sind, so wird  $(x, r)P$  interpretiert als  $(x, r_0)P$ . Zur Lösung der Aufgabe, einen einfachen Kalkül zur Konstruktion möglichst vieler interpretierbarer Sätze aufzustellen, hat der Verf. zwei Systeme entwickelt, von denen eines, das "concealed-type system" dargestellt wird. Jeder Satz des nested-type system ist als eine Interpretation eines Satzes dieses Systems erhältlich.

P. Lorenzen.

Ślupecki, Jerzy. *Le calcul complet de propositions à trois valeurs logiques*. *Ann. Univ. Mariae Curie-Skłodowska*. Sect. F. 1, 193-209 (1946). (Polish. French summary)

This paper concerns two three-valued systems of propositional logic. One of them was previously discussed by the author [*C. R. Soc. Sci. Varsovie. Cl. III. 29, 9-11 (1936)*]. The second system has the peculiarity that two-valued propositional logic is part of it; it has the matrix

<i>C</i>	0	$\frac{1}{2}$	1	<i>N</i>	<i>R</i>
0	1	1	1	1	0
$\frac{1}{2}$	1	1	1	1	1
*1	0	$\frac{1}{2}$	1	0	$\frac{1}{2}$

The matrix for 'C' and 'N' is exactly adequate for two-valued logic. If we add 'R' the system becomes full in the sense that it is possible to define all possible functors of this three-valued logic. The axiomatization reads:  $CCpqCCqrCpr$ ,  $CCNppp$ ,  $CpCNpq$ ,  $CRpNp$ ,  $CRpqRq$ ,  $CpCRqRCpq$ ,  $CRRpp$ ,  $CpRRp$ ,  $NRNp$ . The axiomatization is consistent and complete in the sense that no single propositional variable is a consequence of it, and adding any expression which is not in the system makes a single variable a consequence of it.

H. Hiz (Cambridge, Mass.).

Łoś, Jerzy. Many-valued logics and the formalization of intensional functions. *Kwartalnik Filozoficzny* 17, 59-78 (1948). (Polish)

The purpose of this paper is to present, after an introductory section, an example of a consistent intensional system of logic. This system is an enlargement of the theory of deduction and quantificational logic by addition of a new primitive 'L' such that 'Lxp' has the intuitive inter-

pretation "the man  $x$  believes that  $p$ ." The new axioms read:  $Lx[(p \supset q) \supset [(q \supset r) \supset (p \supset r)]]$ ;  $Lx[p \supset (\sim p \supset q)]$ ;  $Lx[(\sim p \supset p) \supset p]$ ;  $Lx(p \supset q) \supset (Lxp \supset Lxq)$ ;  $Lxp = \sim Lx \sim p$ ;  $LxLxp = Lxp$ ;  $\prod x Lxp \supset p$ . The axioms say that everybody believes the normal theory of deduction, that from two contradictory statements any given person accepts one and rejects the other, that a man believes a given statement if and only if he believes that he believes it, and that any statement believed by everybody is a theorem of the system. The author shows that this system is consistent by means of a model with two persons. This model interprets the system in four-valued logic by classifying statements according as they are believed by both, one or neither of the persons. Nevertheless sentences can still be regarded as either true or false. The system is intensional because, e.g., the expression  $'(p \supset q) \supset [(q \supset p) \supset (Lxp \supset Lxq)]'$  is not a theorem of the system. The main idea in the construction of this intensional system is the use of a function in which a class variable appears together with a sentential variable. The author mentions also an unpublished method presented by S. Leśniewski of avoiding intentional functions, in which an intentional function is regarded as containing not a proposition but a name of a proposition, so that, e.g., 'x believes that  $p$ ' becomes 'x believes in 'p'.'

H. Hiz (Cambridge, Mass.).

Jaśkowski, Stanisław. *Trois contributions au calcul des propositions bivalent*. *Studia Soc. Sci. Torunensis*. Sect. A. 1, 1-15 (1948). (French. Polish summary)

(I) Dans la notation de Łukasiewicz, employant  $A$  (disjonction),  $C$  (implication),  $E$  (équivalence),  $K$  (conjonction),  $N$  (négation), l'auteur donne un système d'axiomes pour le calcul des propositions, comprenant neuf axiomes, dont deux à 9 signes, cinq à 7, un à 6, un à 5 signes. Il démontre que dans tout système suffisant il y a ou bien deux axiomes contenant au moins 9 signes chacun ou bien un axiome à 11 signes. (Le théorème est en effet un peu plus précis.)

(II) Dans l'algèbre de Boole à 2 éléments on peut définir toutes les fonctions à trois arguments par la substitution de 8 fonctions symétriques à sens intuitif simple pour  $F$  dans quelques schémas de la forme  $EFpqGpq$ , où  $G$  est  $A$ ,  $E$ ,  $K$  ou  $C$ , et par une permutation des arguments.

(III) Dans l'algèbre de Boole à 2 éléments  $v, f$ , on considère les fonctions  $F(p_1, \dots, p_n)$ . Si  $F$  prend la valeur  $v$  pour  $m$  systèmes de valeurs des variables, la "probabilité" de  $F$  est  $m \cdot 2^{-n}$ . Les fonctions de probabilité  $2^{-n}$  sont les constituants de Schröder. Soient  $S_i(p_1, \dots, p_n)$  des fonctions données; la substitution  $S(p_i \rightarrow S_i)$  transforme  $F$  en  $S[F]$ ;  $S$  est dit réversible si la substitution inverse de  $S$  existe. A chaque substitution réversible correspond une permutation des constituants et inversement. La probabilité est un invariant pour les substitutions réversibles; il n'existe pas d'invariant indépendant de la probabilité. Application au choix des termes primitifs dans les théories déductives.

A. Heyting (Amsterdam).

Jaśkowski, Stanisław. *Sur les variables propositionnelles dépendantes*. *Studia Soc. Sci. Torunensis*. Sect. A. 1, 17-21 (1948). (French. Polish summary)

On considère les expressions du calcul des prédicts où tous les prédicts ont les mêmes arguments  $x_1, \dots, x_n$  et où toutes les variables  $x_1, \dots, x_n$  sont liées par des quanteurs généraux. La notation de Łukasiewicz est modifiée comme suit: les arguments des prédicts sont omis; le quanteur liant la variable  $x$  est désigné simplement par le signe  $x$  (variable

quantifiante). L'auteur indique un système d'axiomes pour ce calcul  $D$ , qui consiste de (I) un système d'axiomes pour le calcul des propositions, chaque axiome étant précédé par la lettre  $x$ , (III)  $xCyCpqCypyq$ , (II)  $xCypyyp$ , (II)  $xCNypyNyp$ , (II)  $xCyp$ , (II)  $Cxpp$ . Aux règles du calcul des propositions est adjoint le règle: une variable quantifiante peut être remplacée dans toutes ses occurrences par une suite d'une ou de plusieurs variables quantifiantes.

L'auteur énonce sans démonstration quelques théorèmes sur le calcul  $D$ . (1) On peut interpréter dans  $D$  le calcul complet des prédicts. (2) Le problème de décision (Entscheidungsproblem) du calcul des prédicts se réduit au problème de décision dans  $D$  pour les expressions qui ne contiennent qu'une seule variable de prédict et trois variables quantifiantes, et ensuite au problème de décision pour un certain système partiel du calcul des propositions. Soit  $P$  une expression dans  $D$ ; un "facteur" de  $P$  est une variable quantifiante dont  $P$  dépend réellement.  $\text{Fac } PQ$  veut dire:  $Q$  reste vrai pour toutes les valeurs des facteurs de  $P$ ; cette définition peut être remplacée par une définition formelle dans  $D$ . A l'aide du symbole  $\text{Fac}$  on peut définir des modalités; par exemple,  $\text{Fac } pp$  veut dire: "il est nécessaire que  $p$ ."

A. Heyting (Amsterdam).

Jaśkowski, Stanisław. *Sur certains groupes formés de classes d'ensembles et leur application aux définitions des nombres*. *Studia Soc. Sci. Torunensis*. Sect. A. 1, 23-35 (1948). (French. Polish summary)

Étant donné un ensemble  $K$ , les variables  $X, Y, \dots$  désignent des sous-ensembles de  $K$  ("ensembles");  $A, B, \dots$ , des classes d'ensembles ("classes");  $\mathbb{G}$  des familles d'ensembles ("familles"). On dit que  $x \in A$  si  $K - X \in A$ ;  $X \in A \dot{\cup} B$ , s'il existe deux ensembles  $Y, Z$  tels que  $Y \in A$ ,  $Z \in B$  et ou bien  $YZ = 0$ ,  $X = Y + Z$ , ou bien  $Y + Z = K$ ,  $X = YZ$ . On dit que  $\mathbb{G}$  est un "groupe dual," si  $\mathbb{G}$  est un groupe par rapport à  $\dot{\cup}$ , tel que  $\sim A$  est l'inverse de  $A$ , et  $\mathbb{G}$  contient au moins une classe non-vide. Par  $h, k, \dots$  on désigne des fonctions de classes; on définit  $(h+k)(A) = h(A) \dot{\cup} k(A)$ ;  $(-h)(A) = \sim h(A)$ ;  $(h \cdot k)(A) = h(k(A))$ . Si  $A$  est un élément d'un groupe dual quelconque,  $1(A) = A$ ; dans l'autre cas  $1(A)$  est vide. On pose  $0 = 1 + (-1)$ . On dit que  $h$  est un "multiple naturel," lorsque  $h$  est élément de chaque ensemble d'opérations qui contient 1 et qui est clos par rapport à  $+$ . On dit que  $k$  est un "multiple," si  $k = h$  ou  $k = -h$  ou  $k = 0$ , où  $h$  est un multiple naturel. Les multiples satisfont aux axiomes 1-4 de Peano; ils satisfont au cinquième si  $K$  satisfait à l'axiome d'infinité A1: Il existe une classe non-vide  $A$  d'ensembles disjoints et un corps  $C$  d'ensembles (au sens de Hausdorff) tels que  $A \subset C$  et que  $\sum A$  (ensemble des éléments des éléments de  $A$ ) n'est pas élément de  $C$ . On désigne par  $\mathbb{G}[A]$  le seul groupe dual (s'il existe) qui jouit des propriétés suivantes:  $A \in \mathbb{G}$  et  $\sum \mathbb{G}$  est le plus petit parmi les corps d'ensembles  $C$  qui satisfont à  $A \subset C$ ,  $K \in C$ . La classe  $A$  est "positive," si elle contient un élément  $X$  tel que  $X + Y \in K$  pour tout  $Y$  dans  $A$ . On peut maintenant définir les concepts de groupe dual ordonné et de groupe dual continu. Si  $\mathbb{G}[A]$  est continu,  $h/k(A)$  est la classe de tous les  $X$  tels qu'il existe dans  $\mathbb{G}[A]$  un  $B$  contenant  $X$ , pour lequel  $h(A) = k(B)$ ; ici  $h$  est un multiple,  $k$  un multiple naturel. On dit que  $h/k$  est un "coefficient naturel";  $h$  est un "opérateur monotone positif," lorsque pour tout  $A$  pour lequel  $\mathbb{G}[A]$  est continu,  $h(A)$  satisfait aux conditions: si  $A$  est positif,  $h(A)$  est positif; si  $\sim A$  est positif,  $\sim h(A)$  est positif;  $0 \in A$  entraîne  $0 \in h(A)$ . On dit que  $h$  est un "coefficient," lorsque (1) pour chaque  $A$  avec  $\mathbb{G}[A]$

continu,  $h(A) \in \mathbb{G}[A]$ ; (2) pour tout autre  $A$ ,  $h(A)$  est vide; (3) pour tout coefficient rationnel  $k$  on a  $h-k$  où l'un des opérateurs  $h+k$ ,  $k-h$  est monotone positif. L'existence d'un  $A$  avec  $\mathbb{G}[A]$  continu étant supposée (par l'axiome du choix elle suit de A1), les coefficients satisfont aux axiomes des nombres réels. On donne des rapports de cette théorie et de la théorie des fonctions additives d'ensembles.

A. Heyting (Amsterdam).

Nelson, David. Recursive functions and intuitionistic number theory. *Trans. Amer. Math. Soc.* 61, 307-368 (1947).

This paper contains detailed proofs of several results mentioned in a paper by S. C. Kleene [J. Symbolic Logic 10, 109-124 (1945); these Rev. 7, 406]. Using the notation  $\mathcal{R}A$  for the formula representing " $A$  is realizable" we mention the following results. If  $A$  is a formula of  $S_3$ , then  $\mathcal{R}A \rightarrow \mathcal{R} \neg A$  and  $\neg \mathcal{R}A \rightarrow \mathcal{R} \neg \neg A$  are provable in  $S_3$ . For the special primitive recursive function  $T_1(x, y, z)$  the formula  $\mathcal{R}(\forall x(\forall z \neg T_1(x, z) \vee \exists y T_1(x, y, z)))$  is provable in  $S_3$ . The latter is an incompleteness result. Here  $S_3$  denotes a certain formal system of intuitionistic number theory.

A. Heyting (Amsterdam).

Bernays, Paul. A system of axiomatic set theory. VI. *J. Symbolic Logic* 13, 65-79 (1948).

In previous installments [same J. 2, 65-77 (1937); 6, 1-17 (1941); 7, 65-89, 133-145 (1942); 8, 89-106 (1943); these Rev. 2, 210; 3, 290; 4, 183; 5, 198] the author has shown how to develop arithmetic, analysis, elementary set theory and the theory of transfinite numbers without using his sum and power axioms  $V(c)$  and  $V(d)$  or the restrictive axiom VII. The present installment deals with the role of these three axioms in his system. Assuming  $V(c)$  and  $V(d)$  but not VII, he constructs a class II, whose elements are

## ALGEBRA

Lalan, Victor. Étude des coefficients du binôme, modulo le nombre premier  $p$ . *Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iași]* 2, 255-261 (1947).

Write  $(\frac{c}{r})$  as  $b(c, r)$ , and  $c = c_0 + c_1 p + \dots + c_n p^n$ ,  $r = r_0 + r_1 p + \dots + r_n p^n$ ; the result found (proved by induction) is  $b(c, r) = \prod_i b(c_i, r_i) \pmod{p}$ . This is an immediate consequence of the same result for  $n=2$  [see, e.g., E. T. Bell, *Ann. of Math.* (2) 35, 258-277 (1934)].

J. Riordan (New York, N. Y.).

Correnti, Salvatore. Problemi inversi in analisi combinatoria. *Matematiche, Catania* 1, 72-80 (1946).

The problems considered are all related to the following: find  $r$  consecutive integers whose product is a given integer. Conditions for the existence of a solution and rough bounds for the solution when it exists are given.

J. Riordan.

Usai, Giuseppe. Quadri di Tartaglia generalizzati. *Matematiche, Catania* 1, 12-20 (1945).

Certain known properties of multinomial coefficients are developed.

J. Riordan (New York, N. Y.).

Usai, Giuseppe. Su una generalizzazione dei quadri di Tartaglia e su un problema di calcolo delle probabilità. *Matematiche, Catania* 1, 102-103 (1946).

The multinomial coefficients in the note reviewed above are recognized as associated both with a generating func-

called II-sets and whose subclasses are called II-classes. These sets and classes form a subsystem satisfying all his axioms, including VII. This II-model is coextensive with the entire system if and only if the restrictive axiom VII is assumed. Using all his axioms, he shows that every class is of equal power with a class of ordinals, that any two classes are comparable in power, and that every class not represented by a set is of equal power with the class of all sets and with the class of all ordinals. It is also shown that every class, as well as every set, has a well-ordering.

He next constructs four subclasses  $\Pi_0$ ,  $\Pi_1$ ,  $\Pi_2$  and  $\Pi_3$  of the class II, which are proved to be independence models for the axioms VI,  $V(b)$ ,  $V(c)$  and  $V(d)$ . (Axiom VI is the axiom of infinity and  $V(b)$  is the axiom of replacement.) Finally he constructs, as a modification of  $\Pi_0$ , a class  $\Pi^*$  which satisfies all of Zermelo's axioms, and likewise all of Bernays' axioms except  $V(b)$ . In  $\Pi^*$  and hence in Zermelo's system, it cannot be proved that there exists a set of which every finite ordinal is an element. This distinguishes Zermelo's system from the author's.

O. Frink.

\*Beth, E. W. Significs and logic. *Feestbundel Aangeboden door Vrienden en Leerlingen aan H. J. Pos*, pp. 86-95. N. V. Noord-Hollandsche Uitgevers Maatschappij, Amsterdam, 1948.

This is an elementary exposition of "two conceptions of language within the philosophy of science."

C. C. Torrance (Annapolis, Md.).

Devaux, Ph. Dialectique et logique. *Dialectica* 2, 95-108 (1948).  
 Beth, E. W. Les relations de la dialectique à la logique. *Dialectica* 2, 109-119 (1948).  
 Gonseth, F. À propos des exposés de MM. Ph. Devaux et E. W. Beth. *Dialectica* 2, 120-125 (1948).

tion and a probability problem: the appearance of sum  $s$  in a throw of  $r$   $n$ -faced dice.

J. Riordan.

Chakrabarti, S. Some identities and recurrences. *J. Indian Math. Soc. (N.S.)* 11, 89-94 (1947).

Four sum (identities) and three determinant (recurrents) evaluations, too involved to quote, expressing relations of the elementary symmetric functions of the quantities  $1, a, a^2, \dots, a^{n-1}$ . These functions appear as generating functions in the theory of partitions of numbers [the author's first formula, which he takes from a 1923 paper of his own, appears in MacMahon, *Combinatory Analysis*, Cambridge University Press, 1916, v. 2, p. 66] but no hint is given of the possible use of the results in this field or elsewhere.

J. Riordan (New York, N. Y.).

Tenca, Luigi. Sul risultante delle due equazioni  $a_1 + a_2 x + a_3 x^2 + \dots + a_n x^{n-1} = 0$  e  $x^n - 1 = 0$ . *Period. Mat. (4)* 26, 51-55 (1948).

The resultant of a polynomial  $f$  of degree  $n-1$  and the polynomial  $x^n - 1$  is expressed as a determinant of order  $n$ , namely, the cyclic determinant  $C$  formed from the coefficients of  $f$ . The sequence of subresultants  $R_i$  can be generated by taking  $R_0 = C$  and forming  $R_i$  by omitting the last row and column from  $R_{i-1}$ .

J. M. Thomas.

**Amante, Salvatore.** Sulle matrici ortogonali periodiche razionali e in particolare su quelle di 3° ordine cayleyane. *Matematiche, Catania* 2, 10–36 (1947).

There exist proper orthogonal matrices of order 3 with rational elements which are periodic of period  $m$  if and only if  $m=1, 3, 4$  or 6. In each case a matrix  $A$  whose elements are rational numbers is found such that all rational matrices of that period are expressible in the form  $TAT^{-1}$ , where  $T$  is rational and orthogonal but not necessarily periodic.

C. C. MacDuffee (Madison, Wis.).

**Abelès, Florin.** Sur l'élévation à la puissance  $n$  d'une matrice carrée à quatre éléments à l'aide des polynomes de Tchébychev. *C. R. Acad. Sci. Paris* 226, 1872–1874 (1948).

Let  $A$  be a two-rowed square matrix of trace  $p$  and determinant  $\Delta \neq 0$ . Let  $S_n, C_n$  be Chebyshev polynomials and  $U_n, V_n$  functions of Lucas. Then  $A^n = U_n(p, \Delta) \cdot A - \Delta U_{n-1}(p, \Delta) \cdot I$ ,  $U_n(p, \Delta) = \Delta^{(n-1)/2} S_{n-1}(p\Delta^{-1})$ ,  $V_n(p, \Delta) = \Delta^{1/2} C_n(p\Delta^{-1})$ . Thus the functions of Lucas may be calculated from tables of the Chebyshev polynomials.

C. C. MacDuffee.

**Maruashvili, T. I.** On the roots of determinants determining the critical stress. *Bull. Acad. Sci. Georgian SSR [Soobščenija Akad. Nauk Gruzinskoi SSR]* 7, 103–111 (1946). (Russian)

For a class of determinants which arise in the theory of stability of beams it is shown, by using Sturm's chains, that all eigenvalues are real and nonnegative.

M. Kac.

**Parker, W. V.** Characteristic roots and the field of values of a matrix. *Duke Math. J.* 15, 439–442 (1948).

If  $A = (a_{ij})$ ,  $i, j = 1, \dots, n$ , is a square matrix over the complex number field, the set of complex numbers  $xAx'$ , where  $x$  is a row vector of unit norm, is called the field of values of  $A$ . The author proves several theorems on the field of values of  $A$  including: (a) the field of values of every principal sub-matrix of  $A$  is a subset of the field of values of  $A$ ; (b) the field of values of  $A$  is identical with the set of all diagonal elements of the unitary transforms of  $A$ . He also shows that if

$$m = n^{-1} \sum_{i=1}^n a_{ii}, \quad P_k = \sum_{j=1}^n a_{kj}, \quad Q_k = \sum_{i=1}^n a_{ik},$$

where  $\sum'$  omits the terms with equal subscripts, and

$$R_k = P_k + |a_{kk} - m|, \quad T_k = Q_k + |a_{kk} - m|,$$

then every characteristic number of  $A$  lies in or on the circle with center  $m$  and radius  $\min(R, T)$ , where  $R$  and  $T$  are the greatest values of  $R_k$  and  $T_k$ , respectively,  $k = 1, \dots, n$ .

J. Williamson (Flushing, N. Y.).

### Abstract Algebra

**Schwan, W.** Perspektivitäten in allgemeinen Verbänden. *Math. Z.* 51, 126–134 (1948).

A "Teilbund"  $T$  is a subset of the elements of a lattice  $V$  under the same ordering relation; even if  $T$  is a lattice, it need not be a sublattice of  $V$ . An element  $z$  of  $V$  is said to be progressively decomposed with respect to elements  $a$  and  $b$  of  $V$  if  $z = x \vee y$  with  $x \leq a$  and  $y \leq b$ ; in the dual case the decomposition is called regressive. The set of  $z$  in  $V$  which can be regressively decomposed with respect to  $a$  and  $b$  and such that  $z \leq a$  is denoted by  $a/(a \wedge b)$ ;  $(a \vee b)/b$  is defined

dually. The sublattice of  $V$  consisting of all elements of  $V$  between  $a \wedge b$  and  $a$ , denoted by  $a/(a \wedge b)$ , contains  $a/(a \wedge b)$ . The modular law is shown to be equivalent to the condition  $a/(a \wedge b) = a/(a \wedge b)$  for all  $a$  and  $b$ . If  $a$  and  $b$  in  $V$  are given, then the correspondence  $x = a \wedge y \leftrightarrow y = b \vee x$  is called a perspectivity. The sets  $X = a/(a \wedge b)$  of all such  $x$  and  $Y = (a \vee b)/b$  of all such  $y$  are then isomorphic under this correspondence. P. M. Whitman (Silver Spring, Md.).

**McKinsey, J. C. C.** On the representation of projective algebras. *Amer. J. Math.* 70, 375–384 (1948).

Everett et Ulam [même *J.* 68, 77–88 (1946); ces Rev. 7, 409] ont défini une classe particulière d'algèbres booléennes, les algèbres projectives, où sont données deux opérations de "projection"  $a \rightarrow a_x, a \rightarrow a_y$ , satisfaisant à des conditions déduites de l'étude d'un modèle particulier d'algèbre projective, l'algèbre booléenne des parties d'un ensemble produit  $X \times Y$ ; ils ont démontré en outre qu'une algèbre projective atomique et complète (c'est-à-dire formée de tous les sous-ensembles d'un ensemble) est isomorphe à une sous-algèbre de l'algèbre de toutes les parties d'un produit. L'auteur étend ce résultat à toutes les algèbres projectives; sa méthode consiste à définir dans l'algèbre booléenne des ensembles d'idéaux premiers de l'algèbre  $A$  donnée deux opérations de "projection" vérifiant les axiomes d'Everett-Ulam, et de montrer que  $A$  peut être plongée dans l'algèbre projective atomique et complète ainsi définie; la conclusion découle alors du théorème d'Everett-Ulam.

J. Dieudonné (Nancy).

**Cohn, Richard M.** Manifolds of difference polynomials. *Trans. Amer. Math. Soc.* 64, 133–172 (1948).

The author studies systems of abstract difference polynomials from the point of view of the Ritt school [cf. J. F. Ritt and H. W. Raudenbush, same *Trans.* 46, 445–452 (1939); these Rev. 1, 101]. Part I. If  $F \subseteq G \subseteq H$  are difference fields and  $\alpha$  in  $H$  satisfies an algebraic difference equation over  $G$  then  $\alpha$  is "transformally algebraic" over  $G$ . Lemma: if  $\alpha$  is transformally algebraic over  $G$  and each  $\gamma$  in  $G$  is transformally algebraic over  $F$  then  $\alpha$  is transformally algebraic over  $F$ . This lemma [whose proof is not easy] leads to the concept of "degree of transformal transcendence" and, together with the analogue of "generic zero," to the analogue of "dimension" for a prime reflexive difference ideal (of difference polynomials).

Part II. Let  $A$  be a difference polynomial in  $y$  of order  $m$ , degree  $r$  in the  $m$ th transform of  $y$ , and effective order (order minus order of lowest transform of  $y$  present in  $A$ )  $n$ . The author studies the way the perfect difference ideal generated by  $A$  decomposes into prime reflexive difference ideals (none containing another):  $\{A\} = \Pi_1 \cap \dots \cap \Pi_s$ . Theorem: at least 1 and at most  $r$  of the  $\Pi_i$ 's fail to contain a difference polynomial of effective order less than  $n$ . The manifolds of these  $\Pi_i$ 's are the "ordinary" manifolds of  $A$ ; those of the other  $\Pi_i$ 's are the "singular" manifolds of  $A$ . Theorem: if  $A$  is of order 1 there are no singular manifolds.

Part III. A prime reflexive difference ideal  $\Pi$  of difference polynomials in  $y_1, \dots, y_n$  over  $F$  is "quasi-linear" if no extension of  $F$  contains more than one solution of  $\Pi$ . If  $\Sigma$  is a prime reflexive difference ideal in the unknowns  $u_1, \dots, u_q, y_1, \dots, y_p$  ( $u_1, \dots, u_q$  being the arbitrary unknowns) over  $F$ , and if  $q > 0$  or  $F$  contains an element not equal to its transform, then there exists a prime reflexive difference ideal  $\Pi$  (a "resolvent system" for  $\Sigma$ ) in unknowns  $u_1, \dots, u_q, w$  such that, roughly speaking, the manifold of

$\Pi$  is a rational image of the manifold of  $\Sigma$ , and the latter is a quasi-linear image of the manifold of  $\Pi$ .

E. R. Kolchin (New York, N. Y.).

**Deuring, Max.** Zur Theorie der elliptischen Funktionenkörper. *Abh. Math. Sem. Univ. Hamburg* 15, 211–261 (1947).

An algebraic function field of one variable  $K$  with the coefficient field  $k$  is called elliptic if (i) it is separably generated and has genus one and (ii) the genus is preserved for any extension of the coefficient field. Let  $\bar{k}$  denote the algebraic completion of  $k$  and suppose that  $\bar{R} = \{\mu, \dots\}$  is the ring of (normalized) meromorphisms of  $K\bar{k}$ . If  $k(x, y) = K$  with a normalized equation  $f(x, y) = 0$  (depending on the characteristic  $p$  of  $k$  and the value of the absolute invariant  $j$  of  $K\bar{k}$ ), the author determines for a meromorphism  $\mu$  the field of definition  $k_\mu$ , that is, the smallest (separable) extension of  $k$  such that the transforms  $x^\mu$  and  $y^\mu$  lie in  $k_\mu(x, y)$ . The author's computations are carried out in complete detail, and are based upon the structure of the (finite) unit group of  $\bar{R}$ ; the explicit method consists in expressing the invariant  $j$  of the elliptic field under consideration in terms of the coefficients of corresponding normalized equations; e.g., for a unit  $\mu$  and  $p \neq 2, 3$  with  $j \neq 0$ ,  $2^p 3^2$  one has to consider  $k_\mu = k(a)$ , where  $a^2 = g_2 g_3^{-1} g_1^{-1} g_3$  with  $y^2 = 4x^3 - g_2 x - g_3$  and  $(y^\mu)^2 = 4(x^\mu)^3 - g_{1\mu} x^\mu - g_{3\mu}$ . In this fashion the author also determines the extension  $k'$  of  $k$  such that an isomorphism between  $K\bar{k}$  and  $K_1\bar{k}$  ( $K$  and  $K_1$  elliptic over  $k$ ) can be expressed rationally over  $k'$ . It turns out that  $[k':k]$  is a divisor of the number of units in  $\bar{R}$ , provided  $K$  and  $K_1$  contain prime divisors of degree one. This result depends upon a kind of Galois correspondence between certain subfields of  $\bar{k}/k$  and certain subrings of  $\bar{R}$ . Each extension  $k_1$  of  $k$  determines for  $Kk_1$  the set  $R(k_1)$  of meromorphisms  $\mu \in \bar{R}$  which are rationally expressible in  $Kk_1$ , and conversely each subset  $R_1 \subseteq \bar{R}$  determines a subfield

$$k\{R_1\} = \bigcup_{\mu \in R_1} k_\mu$$

of  $\bar{k}$ . Then  $R(k_1) \cup R(k_2) = R(k_1 \cup k_2), \dots, k\{R_1\} \cap k\{R_2\} = k\{R_1 \cap R_2\}$ . It is then shown that  $k^* = k\{\bar{R}\}$  is a finite normal extension of  $k$  whose Galois group  $G = \{\sigma, \dots\}$  determines on  $\bar{R}$  a group of automorphisms by the relation  $\mu \rightarrow \sigma \mu \sigma^{-1}$  which is an isomorphism between  $R(k_\mu)$  and  $R(k_{\mu^*})$ , and thus  $G$  has a crossed representation in the unit group of  $\bar{R}$ . Detailed computations indicate precisely which groups  $G$  and extensions of  $k$  can occur; the results are obtained for the various combinations of values for the characteristic  $p$  and the absolute invariant. For imaginary quadratic  $\bar{R}$  and quadratic  $k^*/k$ ,  $p=0$ ,  $\mu \rightarrow \sigma \mu \sigma^{-1}$  means the passage to the conjugate imaginary in  $\bar{R}$ , and for  $p \neq 0$  always  $G=1$  with  $R(k) = \bar{R}$ . The cases in which  $\bar{R}$  is the maximal order of a rational quaternion algebra (ramified at  $p$  and at infinity), present serious computational handicaps for the determination of the complete correspondence between subrings of  $\bar{R}/R(k)$  and subfields of  $k^*/k$ ; it is shown in detail precisely which subrings and subfields do occur and correspond to each other.

O. F. G. Schilling (Chicago, Ill.).

**Dieudonné, Jean.** Complément à mon article "Sur les corps ordonnables." *Bol. Soc. Mat. São Paulo* 2, 35 (1947).

The author rectifies an omission in one of the proofs in the paper cited [same Bol. 1, 69–75 (1946); these Rev. 9, 77].

I. Kaplansky (Princeton, N. J.).

**Dieudonné, Jean.** Sur les extensions transcendantes séparables. *Summa Brasil. Math.* 2, no. 1, 1–20 (1947).

An extension  $E$  of a field  $K$  of characteristic  $p$  is separable if every subset  $X$  of  $E$  linearly independent over  $K$  has the set  $X^p$  of its  $p$ th powers linearly independent over  $K$ . This is equivalent to the assertion that the fields  $E$  and  $K^{p-1}$  are linearly disjoint relative to  $K$ , in the sense of Weil [Foundations of Algebraic Geometry, Amer. Math. Soc. Colloquium Publ., v. 29, New York, 1946; these Rev. 9, 303]. These concepts are used to prove the theorem of Chevalley, that a separable extension of a separable extension is separable, and to improve and extend a number of results found by the reviewer [Duke Math. J. 5, 372–393 (1939)]; in particular, an extension  $E$  is separable over  $K$  if and only if  $p$ -independent subsets of  $K$  remain  $p$ -independent in  $E$ . In addition, it is shown that an extension  $E$  generated over  $K$  by a finite number of elements contains a (not necessarily unique) maximal separable subfield  $F$  which has a separating transcendence basis over  $K$ , and is such that  $E$  is a subfield of  $K^{p^\infty}(F)$ . If  $m$  is the number of elements in a relative  $p$ -basis of  $F(E^p)$  over  $F$ , the sum  $f$  of all nonzero  $m$  gives the degree  $p^f$  of  $E$  over  $F$ , and is the invariant of  $E/K$  called the order of inseparability. Certain properties of this order, due to Weil [op. cit.] are deduced. If  $E$  has a separating transcendence basis over  $K$ , then  $E^{p^\infty}$  is algebraic over  $K^{p^\infty}$  [but not necessarily contained in  $K^{p^\infty}$ , as incorrectly stated by the reviewer, op. cit.].

S. MacLane (Chicago, Ill.).

**MacLane, Saunders.** Symmetry of algebras over a number field. *Bull. Amer. Math. Soc.* 54, 328–333 (1948).

Let  $A$  be a simple algebra,  $K$  its center,  $Q$  a group of automorphisms of  $K$ , and  $k$  the field of elements left fixed by  $Q$ . Then  $A$  is called  $Q$ -normal if the automorphisms in  $Q$  can all be extended to automorphisms of  $A$ . If  $A$  is the Kronecker product (over  $k$ ) of  $K$  and some simple algebra with center  $k$ , then  $A$  is called trivially  $Q$ -normal. It was proved by Teichmüller [Deutsche Math. 5, 138–149 (1940); these Rev. 2, 122] and by Eilenberg and MacLane [see the following review] that, for arbitrary  $K$ , every  $Q$ -normal  $A$  is trivially so when  $Q$  is cyclic. If  $K$  is a number field, then by use of the invariants of  $A$  it is proved here that the group of  $Q$ -normal  $A$  modulo the group of trivially such is cyclic of order  $s$ , where  $s$  is the greatest common divisor, over all prime divisors  $p$  of  $k$ , of the numbers  $S(p)$  of prime factors of  $p$  in  $K$ . If  $Q$  is cyclic,  $s=1$  but  $s$  can differ from 1 if  $Q$  is, say, the four-group. G. Whaples (Bloomington, Ind.).

**Eilenberg, Samuel, and MacLane, Saunders.** Cohomology and Galois theory. I. Normality of algebras and Teichmüller's cocycle. *Trans. Amer. Math. Soc.* 64, 1–20 (1948).

On considère des extensions galoisiennes (i.e., séparables et normales) et finies d'un corps commutatif  $P$ . Pour chaque extension  $N$ , soit  $G_N$  le groupe de Galois de  $N$  sur  $P$ . On envisage les groupes de cohomologie  $H^n(G_N, N^*)$  du groupe (fini)  $G_N$  avec coefficients dans le groupe (abélien) multiplicatif  $N^*$  des éléments non nuls de  $N$ , sur lequel opère  $G_N$ . Si  $N \subset K$ , il existe un homomorphisme canonique  $H^n(G_N, N^*) \rightarrow H^n(G_K, K^*)$ . On désigne par  $H_0^n(G_N, N^*)$  la réunion des noyaux de ces homomorphismes pour tous les  $K$  possibles. Les auteurs étudient spécialement les groupes  $H^n(G_N, N^*)$  et  $H_0^n(G_N, N^*)$  pour  $n=1, 2, 3$ . Le groupe  $H^n$  est trivial [théorème de Speiser, Math. Z. 5, 1–6 (1919)]; de même le groupe  $H_0^n$  (trivial: se réduit à l'élément neutre).

Le principal théorème du mémoire se rapporte aux travaux de Teichmüller [Deutsche Math. 5, 138-149 (1940); ces Rev. 2, 122] qui sont interprétés en termes de  $H^1$  et complétés: appellen "normale sur  $P$ " toute algèbre simple  $A$  de centre  $N$  telle que tout automorphisme de  $G_N$  se prolonge en un automorphisme de  $A$ ; cette propriété ne dépend que de la classe de  $A$  sur  $N$ , et les classes d'algèbres normales sur  $P$  forment un sous-groupe du groupe des classes (groupe de Brauer). Parmi ces classes figurent les classes (sur  $N$ ) des algèbres obtenues à partir d'une algèbre simple de centre  $P$  par extension (de  $P$  à  $N$ ) du corps des scalaires (mais les algèbres d'une telle classe ne sont pas toutes obtenues par extension du corps des scalaires). Les résultats de Teichmüller impliquent: chaque algèbre simple sur  $N$ , normale sur  $P$ , définit un élément de  $H^1(G_N, N^*)$  qui ne dépend que de la classe de l'algèbre; au produit des classes correspond le composé des éléments correspondants de  $H^1$ ; le noyau de l'homomorphisme ainsi défini se compose des classes des algèbres obtenues par extension du corps des scalaires. Les auteurs complètent les résultats de Teichmüller par le suivant: l'homomorphisme défini plus haut applique le groupe des classes exactement sur le sous-groupe  $H^1(G_N, N^*)$ .

Les démonstrations et les résultats sont parallèles à ceux relatifs au problème des extensions de groupes (non abéliens) [cf. Eilenberg et MacLane, Ann. of Math. (2) 48, 326-341 (1947); ces Rev. 9, 7]. Un lien précis est établi entre les deux catégories de problèmes.

H. Cartan (Paris).

**Levi, F. W.** On skewfields of a given degree. J. Indian Math. Soc. (N.S.) 11, 85-88 (1947).

Let  $a_1, a_2, \dots, a_r$  be elements of an associative ring  $R$ , and set  $[a_1, \dots, a_r] = \sum \pm (a_i a_{i_1} \dots a_{i_r})$ , where the sum extends over the  $r!$  permutations of  $1, 2, \dots, r$ , with the usual convention as to sign. If  $[a_1, \dots, a_r] \neq 0$  for some elements  $a_1, \dots, a_r$ , but  $[a_1, \dots, a_r] = 0$  for all  $a_1, \dots, a_r$ , with  $r > r$ , then  $r$  is said to be the "rank of commutativity" of  $R$ . The author has studied the implications of this notion in another paper [to appear elsewhere] and has shown that for central simple algebras the rank of commutativity depends only on their degree and the characteristic of the ground field. In the present note he remarks that the rank of commutativity, and hence the degree, of a division algebra may be determined by means of relations involving only elements belonging to the commutator subgroup of the multiplicative group of the algebra, independently of a knowledge of the centre or of a particular basis. [Misprints: in formula (4) for  $=$  read  $\equiv$ ; in formulae (9) and (12) read  $-$  for the second and third  $=$  signs, respectively.]

S. A. Jennings (Vancouver, B. C.).

**Fuchs, Ladislas.** On relatively primary ideals. Norske Vid. Selsk. Forh., Trondhjem 20, no. 7, 25-28 (1947).

In a Noetherian integral domain, an ideal  $G$  is defined to be relatively primary to an ideal  $F$  if  $F:G$  is contained in the radical of  $F$ . It is proved that  $G$  is relatively primary to  $F$  if and only if it is contained in no isolated primary component of  $F$ . Also, an intersection-reducible ideal is an intersection of ideals relatively primary to one another.

I. S. Cohen (Cambridge, Mass.).

**Fuchs, Ladislas.** Further generalization of the notion of relatively prime ideals. Bull. Calcutta Math. Soc. 39, 143-146 (1947).

Let  $R$  be a Noetherian integral domain. If  $G, H, J$  are ideals in  $R$ ,  $G$  is said to be relatively prime to  $H$  with respect

to  $J$  if  $H:G \subseteq J$ . (If  $J = H$ ,  $G$  is relatively prime to  $H$  in the ordinary sense.) Let  $\phi$  be a mapping of the set of ideals of  $R$  into itself such that  $H \subseteq \phi(H)$ ,  $\phi(H_1 \cap H_2) = \phi(H_1) \cap \phi(H_2)$ . The main theorem states that if  $H = \cap_{i=1}^n H_i$ , then under certain conditions  $G$  is relatively prime to  $H$  with respect to  $\phi(H)$  if and only if  $G$  is relatively prime to certain of the  $H_i$  with respect to  $\phi(H_i)$ . The case when  $\phi(H)$  is the radical of  $H$  gives the main result of a previous paper of the author [cf. the preceding review].

I. S. Cohen.

**Leavitt, William G.** A normal form for matrices whose elements are holomorphic functions. Duke Math. J. 15, 463-472 (1948).

Let  $\mathfrak{R}$  be a principal ideal ring. [The author considers only the case that  $\mathfrak{R}$  is the ring formed by all holomorphic functions in a closed bounded domain. To establish that it is a principal ideal ring, one can use the fact that every element of  $\mathfrak{R}$  is associated with a polynomial.] A matrix is said to be unimodular if its determinant is a unit of the ring. Then every matrix with characteristic roots in the ring is similar under the group of unimodular matrices to a matrix with zeros above the main diagonal. The reviewer suggests the following simpler proof. Let  $\gamma$  be a characteristic root of a matrix  $A$ . Since  $\gamma$  belongs to  $\mathfrak{R}$ , we have a vector  $\xi$  of  $\mathfrak{R}$ , such that  $\gamma A = \gamma \xi$ . Since  $\mathfrak{R}$  is a principal ideal ring, we may assume that the greatest common divisor of the components of the vector is 1. It is well known [MacDuffee, The Theory of Matrices, Ergebnisse der Math., v. 2, no. 5, Springer, Berlin, 1933, theorem 21.1] that we have a unimodular matrix  $T$  with  $\xi$  as its first row. Then  $(1, 0, \dots, 0) T A T^{-1} = \gamma (1, 0, \dots, 0)$ . That is,  $T A T^{-1}$  is a matrix with  $(\gamma, 0, \dots, 0)$  as its first row. Proceeding in the same way we have the theorem.

L.-K. Hua.

**Brown, Bailey, and McCoy, Neal H.** The radical of a ring. Duke Math. J. 15, 495-499 (1948).

Elementary proofs are given for the main properties of the radical of a ring as defined by the authors in a previous paper [Amer. J. Math. 69, 46-58 (1947); these Rev. 8, 433]. A new characterization of this radical is given.

C. Chevalley (Paris).

**Smiley, M. F.** The radical of an alternative ring. Ann. of Math. (2) 49, 702-709 (1948).

The radical of Perlis-Jacobson (the union of all quasi-regular right ideals) is carried over to alternative rings. It retains many of its important properties: notably, it is a two-sided ideal and left-right symmetric. On the other hand, the connection with maximal ideals and structure theory remains undetermined. The proofs rest on devices for proving associativity when needed, for which purpose the author shows in an appendix that any three elements which associate generate an associative subring. These results are more general than some of those in a paper of Dubisch and Perlis which appeared at the same time [see the following review].

I. Kaplansky (Princeton, N. J.).

**Dubisch, Roy, and Perlis, Sam.** The radical of an alternative algebra. Amer. J. Math. 70, 540-546 (1948).

The authors consider four definitions of the radical of an alternative algebra  $\mathfrak{A}$ . These are: (a) Zorn's original definition as the set  $\mathfrak{R}$  of all properly nilpotent elements of  $\mathfrak{A}$  [Ann. of Math. (2) 42, 676-686 (1941); these Rev. 3, 100], (b) the extension to the alternative case of Jacobson's radical  $\mathfrak{Q}$ , defined as the union of all quasi-regular right ideals [same J. 67, 300-320 (1945); these Rev. 7, 2],

(c) Albert's definition of the radical of any nonassociative algebra  $\mathfrak{A}$  as the intersection  $\mathfrak{S}$  of all ideals  $\mathfrak{B}$  such that  $\mathfrak{A}-\mathfrak{B}$  is the direct sum of simple algebras [Bull. Amer. Math. Soc. 48, 891-897 (1942); these Rev. 4, 130] and (d) Perlis's definition of the radical as the set  $\mathfrak{N}$  of all elements  $h$  such that  $g+h$  is regular for all regular  $g$  [Bull. Amer. Math. Soc. 48, 128-132 (1942); these Rev. 3, 264]. The last definition only applies to algebras with an identity.

It is shown that, if  $\mathfrak{A}$  has an identity, then  $\mathfrak{N}$  is an ideal of  $\mathfrak{A}$ , and that the  $\mathfrak{N}$ -radical of  $\mathfrak{A}-\mathfrak{N}$  is zero. Further, the  $\mathfrak{N}$ -,  $\mathfrak{D}$ -, and  $\mathfrak{S}$ -radicals of  $\mathfrak{A}$  all coincide with  $\mathfrak{N}$ . In particular, this yields a particularly simple proof that  $\mathfrak{N}$  is an ideal. Finally, the authors consider the first three definitions in the case of an algebra  $\mathfrak{A}^0$  without an identity, and show that, if we adjoin an identity  $e$  to  $\mathfrak{A}^0$ , the resulting algebra  $\mathfrak{A}$  has the same radical as  $\mathfrak{A}^0$  with any of the three definitions. Hence the three definitions of the radical give the same radical for all alternative algebras.

D. Rees.

**Kaplansky, Irving.** Rings with a polynomial identity. Bull. Amer. Math. Soc. 54, 575-580 (1948).

Soit  $A$  une algèbre sur un corps  $F$ ,  $F[x_1, x_2, \dots, x_r]$  l'algèbre des polynomes non commutatifs à  $r$  indéterminées  $x_1, \dots, x_r$  (ne permuant pas entre elles) à coefficients dans  $F$ . L'auteur dit que  $A$  satisfait une identité polynomiale s'il existe un élément  $f \neq 0$  de  $F[x_1, \dots, x_r]$  tel que  $f(a_1, \dots, a_r) = 0$  quels que soient les  $a_i$  dans  $A$ . Il montre d'abord que si  $A$  est en outre supposée primitive au sens de Jacobson,  $A$  est nécessairement de rang fini sur son centre; utilisant un lemme de Jacobson, on en déduit aussitôt le résultat de ce dernier [Ann. of Math. (2) 46, 695-707 (1945); ces Rev. 7, 238] montrant que toute algèbre primitive dont tout élément est de degré borné sur le centre est de rang fini sur son centre. Enfin, généralisant un résultat de Levitzki [même Bull. 52, 1033-1035 (1946); ces Rev. 8, 435], l'auteur prouve que si  $A$  satisfait une identité polynomiale et si tous ses éléments sont nilpotents,  $A$  est localement finie, c'est-à-dire que toute sous-algèbre de  $A$  engendrée par un nombre fini d'éléments est de rang fini sur son centre.

J. Dieudonné (Nancy).

**Kaplansky, Irving.** Dual rings. Ann. of Math. (2) 49, 689-701 (1948).

Soit  $A$  un anneau topologique,  $I$  un ensemble d'éléments de  $A$ ,  $L(I)$  et  $R(I)$  les annulateurs de  $I$  à gauche et à droite; l'auteur dit que  $A$  est un anneau dual si pour tout idéal à gauche (respectivement à droite) fermé  $I$ , on a  $L(R(I)) = I$  (respectivement  $R(L(I)) = I$ ). Dans une première partie, l'auteur examine la structure des anneaux duals qui sont semi-simples (au sens de Jacobson) et satisfont à la condition un peu plus forte que l'intersection des idéaux à droite réguliers maximaux fermés est réduite à 0. Il montre alors que dans un tel anneau  $A$  il existe un sous-anneau partout dense  $B$  qui est somme directe (infinie) d'anneaux duals simples  $S_i$ . Chacun de ces derniers contient un sous-anneau partout dense formé de matrices infinies sur un corps (non commutatif), n'ayant chacune qu'un nombre fini d'éléments non nuls. Un anneau semi-simple et compact est anneau dual; il en est de même d'un anneau primitif ayant des idéaux minimaux, quand on prend comme voisinages de 0 les intersections des annulateurs à gauche et des annulateurs à droite des éléments du socle de l'anneau primitif.

Dans la deuxième partie, l'auteur étudie les algèbres de Banach qui sont des anneaux duals; il montre que les  $H^*$ -algèbres d'Ambrose [Trans. Amer. Math. Soc. 57, 364-386 (1945); ces Rev. 7, 126] sont des anneaux duals, et en

déduit plus simplement la structure de ces algèbres; enfin, à l'aide de propriétés des algèbres de Banach complètement continues (c'est-à-dire telles que  $x \rightarrow ax$  et  $x \rightarrow xa$  soient complètement continus), il montre que si  $G$  est un groupe compact, l'algèbre  $C$  des fonctions continues réelles sur  $G$  et les algèbres  $L$ , sur  $G$  sont des anneaux duals semi-simples, d'où il déduit des résultats sur les idéaux dans ces algèbres.

Le rapporteur remarque que la fin de la démonstration du théorème 2 [pp. 690-691] est insuffisante; on peut la remplacer par le raisonnement suivant: pour tout  $a \in A$ ,  $y = xaxB$  et  $xaxI = 0$ , donc  $xaxx = 0$  d'après l'hypothèse, d'où  $(xxA)^2 = 0$ , et comme  $A$  n'a pas d'idéaux nilpotents,  $xxA = 0$ , donc  $xx = 0$ . D'autre part, plusieurs des résultats attribués à Jacobson dans la remarque de la p. 695 sont dûs au rapporteur [Bull. Soc. Math. France 70, 46-75 (1942); ces Rev. 6, 144; ce mémoire n'est pas cité dans la bibliographie].

J. Dieudonné (Nancy).

**Arens, Richard F., and Kaplansky, Irving.** Topological representation of algebras. Trans. Amer. Math. Soc. 63, 457-481 (1948).

Les auteurs caractérisent tout d'abord par des propriétés algébriques l'anneau de toutes les fonctions continues définies dans un espace localement compact  $X$  totalement discontinu, prenant leurs valeurs dans un anneau simple discret  $R$  ayant un élément unité, et nulles hors d'un ensemble compact: (1) un tel anneau  $A$  doit être birégulier, c'est-à-dire que tout idéal bilatère principal doit être engendré par un idempotent dans le centre de  $A$ ; (2)  $A$  doit admettre  $R$  comme anneau d'opérateurs à gauche tel que  $1 \cdot x = x$  et  $\lambda(xy) = (\lambda x)y$  (mais en général  $\lambda(xy) \neq x(\lambda y)$ ) pour  $\lambda \in R$ ,  $x$  et  $y$  dans  $A$ ; (3) pour tout idéal maximal  $M$  de  $A$ ,  $A/M$  doit être canoniquement isomorphe à  $R$ . Ce théorème est une généralisation du théorème bien connu de Stone sur les anneaux booléiens, l'"espace de structure"  $X$  de  $A$  étant l'ensemble des idéaux maximaux de  $A$ . A l'aide d'une série de lemmes algébriques, les auteurs appliquent ce résultat à la représentation, comme anneaux de fonctions continues, des algèbres  $A$  sur un corps  $K$  qui sont commutatives, semi-simples (au sens de Jacobson) et algébriques (tout élément de  $A$  est algébrique sur  $K$ ); ils montrent que  $A$  est isomorphe à un anneau de fonctions continues définies dans un espace localement compact totalement discontinu  $X$ , à valeurs dans l'extension algébriquement close  $M$  de  $K$ , nulles hors d'un ensemble compact, et satisfaisant en outre à deux conditions: (1) à chaque extension  $L_n$  de  $K$  comprise entre  $M$  et l'extension séparable maxima  $L \subset M$  de  $K$  correspond une partie fermée  $X_n$  de  $X$  telle que  $f(X_n) \subset L_n$  pour toute fonction de l'anneau; (2) le groupe de Galois  $G$  de  $L$  sur  $K$  admet une représentation  $g \rightarrow g^*$  sur un groupe  $G^*$  d'homéomorphismes de  $X$ , et on doit avoir identiquement  $f(g^*(x)) = g(f(x))$  pour toute fonction de l'anneau. Les auteurs montrent qu'on ne peut pas toujours prendre pour  $X$  l'"espace de structure" de  $A$ , et donnent des conditions supplémentaires moyennant quoi ce choix de  $X$  est possible, et en outre la condition relative au groupe  $G$  disparait. Enfin, ils donnent un théorème analogue sur la représentation, par une algèbre de fonctions continues, de certaines algèbres de Banach commutatives admettant un antiautomorphisme involutif.

J. Dieudonné (Nancy).

**Barbilian, D.** Topologisches Kennzeichnen der vollständig reduziblen bzw. irreduziblen reellen Algebren. Acad. Roum. Bull. Sect. Sci. 26, 363-376 (1946).

In two previous papers [Jber. Deutsch. Math. Verein. 50, 179-229 (1940); 51, 34-76 (1941); these Rev. 2, 259;

3, 179], the author studied the projective space  $R_{n-1}$  defined with regard to an arbitrary ring  $O$ . In the present paper, he considers the case that  $O$  is an algebra with a 1-element over the field of real numbers. It is shown that  $R_{n-1}$  is compact if and only if  $O$  is semisimple.

R. Brauer.

Newhouse, Albert. On finite extending groups. *Bull. Amer. Math. Soc.* 54, 561–565 (1948).

This paper is an affirmative answer to a problem proposed by A. A. Albert [Ann. of Math. (2) 43, 685–707 (1942); these Rev. 4, 186], who had raised the question whether to a finite algebra  $\mathfrak{A}$  with unit element extending groups  $\mathfrak{G}$  exist which are not merely permutation groups with respect to a suitable basis of  $\mathfrak{A}$ . (An extending group  $\mathfrak{G}$  for  $\mathfrak{A}$  is a finite group of nonsingular linear transformations on  $\mathfrak{A}$  considered as a linear space of order  $n > 1$  over a field  $\mathfrak{F}$  which leave the unit element of  $\mathfrak{A}$  unaltered.) The elements of such groups have then a matrix representation of the form  $G = \begin{pmatrix} 1 & 0 \\ B & M \end{pmatrix}$ , where  $M$  is an  $(n-1)$ -rowed nonsingular square matrix in  $\mathfrak{F}$ .

The author proves by a straightforward construction of suitable matrices and by investigation of their characteristic and minimum polynomials (which divide  $x^n - 1$  for suitable  $n$ ) that for every algebra  $\mathfrak{A}$  over  $\mathfrak{F}$  of order  $n > 2$  there exist finite extending groups  $\mathfrak{G}$  which are not permutation groups on any basis of  $\mathfrak{A}$ . For  $n = 2$  such extending groups exist if and only if (i)  $\mathfrak{F}$  is of characteristic  $p$  and contains more than two elements, or (ii)  $\mathfrak{F}$  is of characteristic 0 and contains a primitive  $m$ th root of unity for some  $m > 2$ .

K. A. Hirsch (Newcastle-upon-Tyne).

Kuročkin, V. M. On the theory of locally simple and locally normal algebras. *Mat. Sbornik N.S.* 22(64), 443–454 (1948). (Russian)

An algebra is locally normal simple if every finite subset is contained in a normal simple algebra of finite order; locally matricial is similarly defined. Such an algebra is primary if the finite subalgebras are of prime power order. Theorem 1. A primary algebra is either locally matricial or else has the descending chain condition (and so is a matrix algebra of finite order over a division algebra). The proof proceeds by showing that, in the absence of the chain condition, arbitrarily large matrix subalgebras can be built and used to swallow up any division subalgebras. Theorem 2.

Let  $B$  be a maximal  $p$ -primary subalgebra which is a direct factor of  $A$ . Then any countable  $p$ -primary subalgebra of  $A$  is isomorphic to a subalgebra of  $B$ . If  $B$  is countable, and  $C$  is a second countable subalgebra with the same properties as  $B$ , then  $B$  and  $C$  are isomorphic. The proof rests on building the desired isomorphisms stepwise, using properties of finite algebras. An example shows that the hypothesis that  $B$  is a direct factor cannot be dropped. The final theorem is a sharpening of results of Köthe and Kuročkin. Theorem 3. Let an algebra have two decompositions into a direct product of indecomposable algebras. Then if for each prime we unite the factors belonging to that prime, the resulting primary algebras are pairwise isomorphic.

I. Kaplansky (Princeton, N. J.).

Mal'cev, A. I. On the embedding of group algebras in division algebras. *Doklady Akad. Nauk SSSR (N.S.)* 60, 1499–1501 (1948). (Russian)

Let  $G$  be a simply ordered group and  $A$  its group algebra over a field  $F$ , that is, the set of all finite linear combinations of elements of  $G$  with coefficients in  $F$ . The author proves that  $A$  can be embedded in a division algebra  $B$ , and that if  $F$  is ordered, then  $B$  can be ordered so as to preserve the order of  $F$  and  $G$ . The proof rests on the device of taking all formal (well-ordered) power series. The existence of inverses is proved by justifying the expansion  $(1-u)^{-1} = 1 + u + u^2 + \dots$ . There is no reference to the classical stepwise construction of Hahn [Akad. Wiss. Wien, S.-B. IIa. 116, 601–655 (1907)]; though stated for Abelian groups, Hahn's proof works just as well without commutativity or even associativity. Application is made to the group algebra of the free group; the fact that the latter can be ordered is quoted from Shimbireva [Rec. Math. [Mat. Sbornik] N.S. 20(62), 145–178 (1947); these Rev. 8, 563].

I. Kaplansky (Princeton, N. J.).

Levine, Jack. Lie groups of genus one. *Duke Math. J.* 15, 307–311 (1948).

The nullity of a Lie algebra has been defined by M. S. Knebelman [Ann. of Math. (2) 36, 46–56 (1935)] to be the smallest possible number of generators of the algebra, and the genus to be the difference between the order and the nullity. A necessary and sufficient condition for a Lie algebra to be of genus one is obtained in terms of the constants of structure.

C. Chevalley (Paris).

## THEORY OF GROUPS

Picard, Sophie. Un théorème concernant le nombre des bases d'un sous-groupe transitif et primitif, à base du second ordre, du groupe symétrique. *C. R. Acad. Sci. Paris* 227, 254–256 (1948).

If a transitive and primitive permutation group  $G$  on  $n$  symbols is generated by substitutions  $S, T$  such that  $RTR^{-1} = S$ , then  $R^2 = 1$  and  $G$  is said to have a basis of type 2. The total number of such bases of  $G$  is a multiple of  $m$  or of  $m/2$ , where  $m$  is the order of the normaliser of  $G$  in  $S_n$ .

G. de B. Robinson (Toronto, Ont.).

Hua, Loo-keng. Some "Anzahl" theorems for groups of prime power orders. *Sci. Rep. Nat. Tsing Hua Univ.* 4, 313–327 (1947).

The key result is: if the group  $G$  of order  $p^n$  ( $p$  an odd prime) has a cyclic subgroup of index  $p^a$ , then  $(xy)^{p^a} = x^{p^a}y^{p^a}$  for all  $x, y \in G$ . It follows that the elements of  $G$  of order not exceeding  $p^a$  form a (characteristic) subgroup with cyclic

factor group; and that the  $p^a$ th powers of the elements of  $G$  form a cyclic subgroup contained in the centre of  $G$ . The least  $\alpha$  for which  $G$  has a cyclic subgroup of index  $p^\alpha$  is called the rank of  $G$ . If the rank  $\alpha$  of  $G$  is less than  $n - \alpha$ ,  $G$  has exactly  $p^\alpha$  cyclic subgroups of each of the orders  $p^{\alpha+1}, p^{\alpha+2}, \dots, p^{n-\alpha}$ ; and exactly  $p^{2\alpha+1}$  elements of order not exceeding  $p^{\alpha+1}$  ( $0 \leq k \leq n - 2\alpha$ ). For such  $G$ , the ranks of the major subgroups (intersections of maximal subgroups) are determined, whence by use of a known enumeration principle [Zassenhaus, Lehrbuch der Gruppentheorie, Teubner, Leipzig-Berlin, 1937, p. 116] a variety of further "Anzahl" theorems are derived.

P. Hall (Cambridge, England).

Wall, G. E. Finite groups with class-preserving outer automorphisms. *J. London Math. Soc.* 22 (1947), 315–320 (1948).

Burnside [Proc. London Math. Soc. (2) 11, 225–245 (1912)] has given examples of finite groups with outer auto-

morphisms which change each class of conjugates into itself. In the present paper the same property is proved for the group  $L_m$  of permutations  $x \rightarrow ex + r \pmod{m}$  on the integers  $x = 0, 1, \dots, m-1$  provided  $m$  is divisible by 8. In this case it is proved that the group of class preserving outer automorphisms of  $L_m$  is the direct product of the cyclic group of order 2 by the group of inner automorphisms.

R. M. Thrall (Ann Arbor, Mich.).

Kuntzmann, J. Contribution à l'étude des chaînes principales d'un groupe fini. *Bull. Sci. Math.* (2) 71, 155-164 (1947).

In a distributive lattice no two distinct prime quotients which are projective can belong to the same chain. Hence a partial ordering may be defined among the classes of projective quotients in which  $C_1 > C_2$  if in every chain the quotient belonging to  $C_1$  precedes that belonging to  $C_2$ . A finite distributive lattice is uniquely determined by this order relation. This fact is used in the construction of a finite group having an arbitrary finite distributive lattice for its lattice of normal subgroups. The construction is achieved through the concept of the "combiné" of a set of groups. Let  $G_1, \dots, G_n$  be a series of finite groups such that each element of  $G_i$  is identified with an automorphism of each  $G_j$  for  $j < i$ , that is, if  $a \in G_i$ , then  $a \cdot (a_j)$  is a well defined automorphism of  $G_j$ . The combiné of  $G_1, \dots, G_n$  then is the group consisting of all elements  $A_n = (a_n, a_{n-1}, \dots, a_1)$ , where  $a_i \in G_i$ , and multiplication is defined by induction by the equations  $A_1 B_1 = a_1 b_1$ ,

$$A_k B_k = (a_k, A_{k-1})(b_k, B_{k-1}) = (a_k b_k, b_k (A_{k-1}) B_{k-1}),$$

where  $b_k (A_{k-1}) = (b_k(a_1), b_k(a_2), \dots, b_k(a_{k-1}))$ . If all the automorphisms  $b_k(a_i)$  reduce to the identity the combiné becomes the direct product.

Given a distributive lattice defined by an order relation among its classes, assign to each class  $C_i$  a prime number  $p_i$  such that  $p_i$  divides  $p_j - 1$  if  $C_i > C_j$ . The required group, having the given lattice as the lattice of its normal subgroups, is the combiné of the cyclic groups  $G_1, \dots, G_n$  of orders  $p_1, \dots, p_n$ , in which the automorphisms defined by  $G_i$  in  $G_j$  reduce to the identity if and only if  $C_i$  and  $C_j$  are not comparable in the given order relation. [Misprint: p. 164, line 3, for  $p_{j-1}$  read  $p_j - 1$ .] D. C. Murdoch.

Delsarte, S. Fonctions de Möbius sur les groupes abéliens finis. *Ann. of Math.* (2) 49, 600-609 (1948).

For  $x$  a finite Abelian group the author considers functions  $f(x)$  of  $x$  to the complex numbers such that  $f(x)$  depends only on the type of  $x$ . Any two such functions  $f, g$  are composed to give a new function  $f * g = F(x) = \sum_{d/x} f(d)g(x/d)$ , where  $d/x$  denotes that  $d$  is a subgroup of  $x$ ,  $x/d$  is the corresponding quotient group and summation is extended over all subgroups of  $x$ . The author proves that  $f * g = g * f$  and  $(f * g) * h = f * (g * h)$ . Letting the two functions  $\epsilon(x)$ ,  $\delta(x)$  be defined by  $\epsilon(x) = 1$  for all  $x$  and  $\delta(x) = 1$  if  $x$  contains only one element, and 0 otherwise, the author's generalization of the Möbius function is the solution  $\mu(x)$  of the equation (1)  $\mu * \epsilon = \delta$ . Noting that, if  $x$  contains only one element,  $\mu(x) = 1$ , the existence and uniqueness of  $\mu(x)$  is proved by induction on the order of  $x$ . In terms of the generalized  $\mu$ -function, a generalized Möbius inversion formula is easily obtained in the form (2)  $f * \epsilon = F \Leftrightarrow f = F * \mu$ .

Defining a function  $f(x)$  to be multiplicative if  $f(xy) = f(x)f(y)$  ( $xy$  denotes the direct product of  $x$  and  $y$ ) when-

ever the orders of  $x$  and  $y$  are relatively prime, it is shown that  $\mu(x)$  is multiplicative. Thus the problem of computing  $\mu(x)$  is reduced to the case where  $x$  is a prime power Abelian group. This is done first in the case where  $x$  is of type  $(p, \dots, p)$  and of order  $p^n$ ; by induction on  $m$  the formula  $\mu(x) = (-1)^n p^{m(m-1)}$  is obtained. Then noting that in the general case defining (3)  $\mu(d) = (-1)^n p^{1+(n-1)}$  for  $d/x$ ,  $d$  of type  $(p, \dots, p)$  and order  $p^n$ ,  $\mu(d) = 0$  for other  $d/x$ , yields a solution of (1), the uniqueness of this solution shows that (3) together with the multiplicativity gives the generalized Möbius function explicitly.

The inversion formula (2) (together with (3)) is then used to obtain a formula for the number of subgroups of a given type of a prime power Abelian group. This last result was recently obtained independently by Y. Yeh [Bull. Amer. Math. Soc. 54, 323-327 (1948); these Rev. 9, 492].

H. N. Shapiro (New York, N. Y.).

MacLane, Saunders. Groups, categories and duality. *Proc. Nat. Acad. Sci. U. S. A.* 34, 263-267 (1948).

The direct product and the free product of two groups are defined abstractly in terms of homomorphisms, the two definitions being formally deducible one from the other by applying the following "duality rules": invert the direction of each homomorphism, invert the order of all products of homomorphisms, interchange homomorphisms onto with isomorphisms into. The same duality is observed to hold between free Abelian groups and infinitely divisible Abelian groups. The author aims to formulate these and other similar duality relations of group theory axiomatically. This is done by a refinement of the notion of category, originally introduced by Eilenberg and MacLane [Trans. Amer. Math. Soc. 58, 231-294 (1945); these Rev. 7, 109]. A category is a class of entities called "mappings" (e.g., homomorphisms) in which the products of certain pairs of mappings are defined and satisfy certain axioms (conditional existence and associativity of products, existence of "identities"). A bicategory is now defined to be a category with two (dual) distinguished classes of mappings, called injections and projections, which satisfy certain simple additional postulates. A group can be interpreted as a bicategory with one identity, the mappings of the category being the elements of the group. A lattice can be interpreted as a bicategory whose mappings are all injections: the mappings are the pairs  $[a, b]$  of lattice elements such that  $a \triangleright b$ , with product  $[a, b][b, c] = [a, c]$ . The author states that "most of the phenomena of universal algebra and of (axiomatic) group duality have appropriate and simple formulations in terms of lattice-ordered bicategories." Here, lattice-ordered bicategories are a special class of bicategories which include both groups and lattices (interpreted as above).

P. Hall (Cambridge, England).

van der Waerden, B. L. Free products of groups. *Amer. J. Math.* 70, 527-528 (1948).

A simple proof is given for the existence of the free product of a set of groups which avoids the somewhat tiresome discussions of the associative law and of equality which are involved in the previously published accounts. The free product is obtained as a group of permutations which operate on the set of all words (including the empty word) formed from the elements of the given set of groups, with the convention that consecutive letters of a word must belong to different groups and no letter shall be the identity element.

P. Hall (Cambridge, England).

**Beaumont, Ross A.** Rings with additive group which is the direct sum of cyclic groups. Duke Math. J. 15, 367-369 (1948).

Let  $G$  be an additive Abelian group which is the (not necessarily finite) direct sum of cyclic subgroups  $\{u_i\}$ , and let  $t_i$  be the order of  $\{u_i\}$ , where  $t_i=0$  if  $\{u_i\}$  is infinite. Then every element of  $G$  may be written in the form  $\sum a_i u_i$ , where the sum may perhaps be infinite but for all but a finite number of the integers  $a_i$  we have  $a_i=0 \pmod{t_i}$ , congruence modulo 0 being ordinary equality. The following theorem, a generalization of the well-known conditions upon the multiplication constants of an associative algebra, is obtained. The group  $G$  is the additive group of an associative ring  $R$  if and only if multiplication of the elements  $a=\sum a_i u_i$ ,  $b=\sum b_j u_j$  is defined by setting  $ab=\sum_{i,j} a_i b_j g_{ij} u_i$ , where the  $g_{ij}$  are any integers such that (i)  $g_{i0}=0 \pmod{t_i}$  for fixed  $i$ ,  $j$  except for a finite number of values of  $k$ ; (ii)  $\sum_{i,j} a_i b_j g_{ij} = \sum_{i,j} a_i b_j g_{ij} \pmod{t_i}$  for all  $i$ ,  $j$ ,  $k$ ,  $m$ ; (iii)  $g_{i0}=0 \pmod{(t_k/(t_i, t_j, t_k))}$ , where  $(t_i, t_j, t_k)$  is the greatest common divisor of the nonzero  $t$ 's in the symbol if any, and otherwise  $0/(0, 0, 0)=1$ . *S. A. Jennings.*

**Cernikov, S. N.** Addendum to the paper "On the theory of complete groups." Mat. Sbornik N.S. 22(64), 455-456 (1948). (Russian)

This supplementary note [cf. same vol., 319-348 (1948); these Rev. 9, 566] concerns a generalization of the concept of complete group. Let  $\pi$  be a set of natural primes; a group  $G$  is called  $\pi$ -complete provided that, for every natural number  $n$  all the prime factors of which belong to  $\pi$ , the  $n$ th powers of all the elements of  $G$  form a subgroup of  $G$ . The author correlates the new concept with the results of his previous work, noting appropriate modifications and additional theorems, as well as conclusions not remaining valid under the generalization. *R. A. Good.*

**Lyndon, Roger C.** The cohomology theory of group extensions. Duke Math. J. 15, 271-292 (1948).

Die Cohomologiegruppen  $H^n(G, C)$  einer beliebigen Gruppe  $G$  bezüglich der Abelschen Koeffizientengruppe  $C$  wurden von Eilenberg und MacLane [Ann. of Math. (2) 46, 480-509 (1945); diese Rev. 7, 137] und Eckmann [Comment. Math. Helv. 18, 232-282 (1946); diese Rev. 8, 166] eingeführt. Der Verf. untersucht, wie sie für ein direktes Produkt  $G=A \times B$  mit denjenigen der Gruppen  $A$  und  $B$  zusammenhängen. Das Hauptresultat (1) besagt, dass  $H^n(A \times B, C)$  eine absteigende Kette von Untergruppen  $\tilde{H}^0 = H^n(A \times B, C) \supset \tilde{H}^1 \supset \cdots \supset \tilde{H}^n \supset \tilde{H}^{n+1} = 0$  enthält, derart dass für  $k=0, 1, \dots, n$  ein natürlicher Isomorphismus von  $\tilde{H}^k / \tilde{H}^{k+1}$  in eine Faktorgruppe von  $H^n(A, H^{n-k}(B, C))$  existiert. Ist  $A$  oder  $B$  endlich und  $C$  die additive Gruppe der ganzen Zahlen, so handelt es sich sogar um einen Isomorphismus auf  $H^n(A, H^{n-k}(B, C))$ . Auf Grund bekannter Resultate für zyklische Gruppen gestattet dies eine vollständige Bestimmung der Cohomologiegruppen der Abelschen Gruppen mit endlich vielen Erzeugenden. Es wird angegeben, wie sich der Satz (1) auf den Fall übertragen lässt, wo  $G$  nicht direktes Produkt von  $A$  und  $B$ , sondern eine Erweiterung von  $B$  durch  $A$  ist.

*B. Eckmann (Zürich).*

**Haimo, F.** Preservation of divisibility in quotient groups. Duke Math. J. 15, 347-356 (1948).

This is a contribution (mainly negative) towards the problem of an algebraic characterization of those abstract Abelian groups which can be compact groups (in a suitable

topology). It shows that certain additive groups cannot admit topologies which make them compact topological groups. To any element  $g$  of an Abelian group  $G$  the author attaches as divisibility "pattern" the set of all its heights ["Höhenexponenten" as defined by Prüfer, Math. Z. 17, 35-61 (1923)] with respect to the prime numbers. In a factor-group  $G/H$  of  $G$  the heights of the cosets will, in general, be larger than the heights of the corresponding elements in  $G$ . If, however, the subgroup  $H$  of  $G$  has the property that every coset contains at least one element which has the same pattern in  $G$  as the coset in  $G/H$ , then the author calls the subgroup "pattern-true." Direct summands, for example, are pattern-true and so are subgroups of finite order. Pattern-true subgroups and their cosets have been used extensively by Liapin [Rec. Math. [Mat. Sbornik] N.S. 8(50), 205-237 (1940); these Rev. 3, 195] in his characterization of those Abelian groups which can be decomposed into direct sums of groups of rank 1.

This algebraic property of being pattern-true now lends itself to the discussion of topological properties. Among the author's results are the following. All subgroups of a torsion-free group  $G$  are pattern-true in  $G$ , if and only if  $G$  is a direct sum of copies of the additive group of rational numbers. Every countably compact subgroup of a topological group with division closure [the terminology is that of Lefschetz, Algebraic Topology, Amer. Math. Soc. Colloquium Publ., v. 27, New York, 1942, p. 68; these Rev. 4, 84] is pattern-true in  $G$ , from which it follows that every closed subgroup of a countably compact group with division closure  $G$  is pattern-true in  $G$ , and in particular, that every closed subgroup of a compact group  $G$  is pattern-true in  $G$ . A (nontrivial) countably compact group with division closure possesses a nonzero element which has infinite height for all prime numbers with (at most) one exception. Hence if all the nonzero elements of a group  $G$  have patterns with only finite entries, then  $G$  can in no way be made into a compact group. In particular, direct sums of copies of the integers can never be compact groups. (This last result is well known in the countable case.)

The elements of  $G$  which have only infinite entries in their pattern ("completely divisible") obviously form a subgroup of  $G$ . If now  $G$  is a countably compact group with division closure, then the elements which have infinite entries in their pattern at all but a finite number of places ("almost completely divisible") also form a subgroup  $H(G)$ . This subgroup  $H(G)$  generalizes in some respects the group  $F(G)$  of the torsion elements of  $G$ , and indeed contains  $F(G)$ , if  $G$  has elements of finite order. The author shows that, if  $G$  is compact, then  $H(G)$  is at least everywhere dense in  $G$  (or  $H(G)=G$ ), so that in a compact group an arbitrary neighbourhood of any given element must contain an almost completely divisible element. *K. A. Hirsch.*

**Tartakowsky, W.** Sur un représentation du groupe affine. Rec. Math. [Mat. Sbornik] N.S. 19(61), 19-32 (1946). (Russian. French summary)

Let  $f_i(x_1, \dots, x_n) = \sum a_{i, k_1, \dots, k_n} x_1^{k_1} \cdots x_n^{k_n}$  ( $i=1, \dots, m$ ;  $k_1+\dots+k_n=N$ ) be a system of  $m$  forms of degree  $N$  in  $n$  variables  $x_1, \dots, x_n$ . A nonsingular linear transformation on the  $x$ 's induces in a natural way a nonsingular linear transformation on a space of vectors (designated "multitensors") whose coordinates are determinants of order  $m$  in the  $x$ 's, and this correspondence is in fact an (affine) group representation. The author proves that, if  $N \geq 1$ , and if at least one of  $m$  and  $n$  is equal to two, then for any non-

vanishing multitensor there is a linear transformation on the  $x$ 's which induces a transformation which maps the given multitensor into one none of whose coordinates is zero. Furthermore, if both  $m$  and  $n$  exceed two, there exists a multitensor for which the conclusion of this theorem is false.

*I. E. Segal* (Chicago, Ill.).

**Ogasawara, Tōzirō. Almost periodic functions in groups.** *J. Sci. Hiroshima Univ. Ser. A*, 11, 115–123 (1942).

It is known [see A. Weil, *L'intégration dans les groupes topologiques et ses applications*, Actual. Sci. Ind., no. 869, Hermann, Paris, 1940, p. 135; these Rev. 3, 198] that the space  $A$  of almost periodic functions on a group  $G$  is essentially equivalent to the space of continuous functions on an appropriate compact group  $\bar{G}$ . The author proves this directly from the definition of almost periodicity. If  $H$  is the invariant subgroup  $(x; f(a \times b) = f(ab), f \in A)$  the functions  $f \in A$  can be considered as functions on  $G/H$ . The group  $G/H$  is topologized with help of all  $f \in A$ , a neighborhood of the element  $a$  being of the type

$$U_{(f_1, \dots, f_n; \epsilon)}(a) = (x; \rho_f(a, x) < \epsilon, f_i \in A, i = 1, 2, \dots, n),$$

where  $\rho_f(a, b)$  is the translation function

$$\rho_f(a, b) = \sup_{x, y} |f(xay) - f(xby)|.$$

The group  $\bar{G}$  is obtained by completing  $G/H$  (which is a uniform space) in the sense of A. Weil. It is compact, and every  $f \in A$  extends itself to a continuous function on  $\bar{G}$ ; conversely, every continuous function on  $\bar{G}$  is an almost periodic function on  $G$ .

If, in the above procedure,  $A$  is replaced by an arbitrary set  $T$  of almost periodic functions on  $G$ , a group  $\bar{G}$  is obtained which is compact and has the property that every function in the module spanned by  $T$  extends itself to a continuous function on  $\bar{G}$ , and conversely. In the case where  $T$  consists of one almost periodic function this result is due to van Kampen [Ann. of Math. (2) 37, 78–91 (1936)].

The paper also contains a generalization to vector-valued almost periodic functions of the case  $T = A$ , a proof of the existence of the mean for such functions by the use of Haar measure in  $\bar{G}$ , and various remarks on a paper by Bochner [Ann. of Math. 40, 769–799 (1939); these Rev. 1, 110].

*E. Følner* (Copenhagen).

**Kawada, Yukiyosi. On the group ring of a topological group.** *Math. Japonicae* 1, 1–5 (1948).

Soit  $G$  un groupe localement compact,  $L(G)$  l'algèbre de  $G$  sur les réels, c'est-à-dire l'espace des fonctions réelles sommables dans  $G$  pour la mesure de Haar (à droite), où la multiplication est le produit de composition  $(f \times g)(x) = \int_G f(xy^{-1})g(y)dy$ , et la norme  $\|f\| = \int_G |f(x)|dx$ ; dans  $L(G)$  la relation d'ordre  $f \geq 0$  signifie  $f(x) \geq 0$  presque partout. L'auteur démontre que si  $G_1$  et  $G_2$  sont deux groupes localement compacts tels qu'il existe un isomorphisme algébrique  $T$  de  $L(G_1)$  sur  $L(G_2)$ , tel que  $f \geq 0$  entraîne  $Tf \geq 0$ , alors  $G_1$  et  $G_2$  sont isomorphes (topologiquement). L'idée de la démonstration consiste à représenter  $G$  par le groupe des opérateurs de translation  $U_x$  de  $L(G)$  ( $U_x f(x) = f(ax)$  par définition), muni de la topologie forte, puis à caractériser les opérateurs  $\lambda U_x$  ( $\lambda$  scalaire positif) par les propriétés suivantes qui ne font intervenir que la structure algébrique et la structure d'ordre de  $L(G)$ : ce sont les applications linéaires et biunivoques  $U$  de  $L(G)$  sur lui-même, telles que  $f \geq 0$  entraîne  $Uf \geq 0$ , et  $U(f \times g) = (Uf) \times g$ .

*J. Dieudonné* (Nancy).

**Braconnier, Jean. Sur les groupes topologiques localement compacts.** *J. Math. Pures Appl.* (9) 27, 1–85 (1948).

This paper is mainly concerned with the structure of totally disconnected locally compact Abelian groups, and also to some extent with the group of automorphisms of a locally compact group. Most of the results have previously been abstracted and reviewed [Braconnier, C. R. Acad. Sci. Paris 218, 304–305 (1944); 220, 382–384 (1945); Braconnier and Dieudonné, C. R. Acad. Sci. Paris 218, 577–579 (1944); these Rev. 7, 114]. Apart from results previously reviewed, the author introduces the notion of a local direct product of groups, relative to open invariant subgroups of the groups (a modification of and algebraic subgroup of the usual direct product), and shows that every locally compact Abelian group, which together with its dual is totally disconnected, is a local direct product of primary Abelian groups. Every locally compact Abelian group, each of whose elements is of finite order, is the product of a finite number of primary Abelian groups relative to primes  $\{p_i\}$ , and a discrete Abelian group, which is a product of discrete primary Abelian groups relative to primes distinct from the  $p_i$ .

The automorphism group of a locally compact Abelian group, which together with its dual is totally disconnected, is the local direct product of the automorphism groups of its primary Abelian constituents. The automorphism group of a locally compact group which is the product of closed characteristic subgroups is the direct product of the automorphism groups of the factors.

*I. E. Segal.*

**Graev, M. I. Free topological groups.** *Izvestiya Akad. Nauk SSSR. Ser. Mat.* 12, 279–324 (1948). (Russian)

This is a study of the concept due in essentials to A. A. Markoff [Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 9, 3–64 (1945); these Rev. 7, 7] of a free topological group over a completely regular space  $X$ . The definition here departs from the original in associating with  $X$  an arbitrary constant point  $e$  of  $X$  which in any mapping of  $X$  into a topological group  $G$  is required to map into the identity of  $G$ . The free topological groups  $F(x)$  and Abelian group  $A(x)$  are independent, to within isomorphism, of the particular selection of the point  $e$ . The groups resulting from the two definitions are isomorphic when and only when the space  $X$  is not connected. However, by the device of adjoining an isolated point  $e$  to a given space  $X'$  the author obtains a space  $X$  such that the free topological group in the sense of Markoff of  $X'$  coincides with his own  $F(X)$ .

The methods of this paper differ from those of Markoff's work, even where the theorems are analogous. In the proof of the existence of  $F(x)$ , and closely related Abelian  $A(x)$ , the author uses a preliminary step in which  $X$  is assumed metric. In the free discrete group  $F_x$  generated by elements of  $X$  as basis (with  $e$  as identity element) he defines a metric on the set sum of  $X$  and  $X^{-1}$  (inverses in  $F_x$ ) and extends this metric to all of  $F_x$ ; neat use is made of the word-representation of elements of  $F_x$  in terms of the letters of  $X$  and  $X^{-1}$ . Having thus found one topologization of  $F_x$  which induces in  $X$  its given topology, it is possible to conclude from the existence of a partial ordering of the set of all topologies over a group that the desired  $F(X)$  exists. This conclusion is carried over to the general (completely regular)  $X$  through the intermediary of the family of all continuous real valued functions defined on  $X$ . This type of argument is used on several occasions and replaces the device of a multinorm used by Markoff.

It is shown that spaces differing in such fundamental properties as local compactness and first separability may nonetheless give rise to isomorphic free groups. However, connectedness, bicomplexity and compactness are invariant properties under a concept of  $F$ - (and  $A$ -) equivalence. A result associated with the study of connectedness is the theorem that for every  $n \geq 2$  there exists a connected group whose elements are of order  $n$  or less.

If  $X$  is bicomplex, it is shown that the groups  $F(X)$  and  $A(X)$  are normal spaces, complete in the sense of A. Weil. In this connection there is deduced the existence of a topologization of the additive group of integers which is complete but not locally bicomplex. The case of countable compact sets is completely investigated and the associated  $F(X)$  classified.

Concluding sections are concerned with a certain class of free closed subgroups of  $F(X)$  for  $X$  bicomplex. A criterion is developed that a closed subset  $Y$  of  $X$  shall generate such a subgroup, and an example is constructed of a zero-dimensional (moreover, countable and compact) set  $X$  whose  $A(X)$  has closed subgroups that are not free.

L. Zippin (Flushing, N. Y.).

Borel, Armand, et de Siebenthal, Jean. Sur les sous-groupes fermés connexes de rang maximum des groupes de Lie clos. C. R. Acad. Sci. Paris 226, 1662-1664 (1948).

Let  $G$  be a compact Lie group,  $G'$  a closed connected subgroup of  $G$ ,  $Z'$  the center of  $G'$ . Assume that the rank of  $G'$  is the same as the rank of  $G$ . The authors sketch proofs of the following theorems: (1)  $G'$  is the normalizer of  $G$ ; (2) if  $G'$  is maximal, it is the normalizer of some element of  $G$ .

P. A. Smith (New York, N. Y.).

Clifford, A. H. Semigroups containing minimal ideals. Amer. J. Math. 70, 521-526 (1948).

The purpose of this paper is to extend certain results of Suschkewitsch [Math. Ann. 99, 30-50 (1928)] concerning finite semigroups to the infinite case. The kernel  $K$  of a semi-group  $S$  is defined to be the intersection of all its two-sided ideals, providing that this intersection is nonvacuous. Whenever  $K$  exists, it is a simple semigroup without zero. Further, if  $S$  has a minimal left ideal,  $K$  exists and can be given an alternative characterisation as the union of all minimal left ideals. This latter characterisation is that given by Suschkewitsch [loc. cit.] for the "Kerngruppe."

The above results hold if we replace "left" by "right." If  $S$  contains both a minimal left ideal and a minimal right ideal, the structure of  $K$  is further restricted and must be completely simple in the sense used by the reviewer [Proc. Cambridge Philos. Soc. 36, 387-400 (1940); these Rev. 2, 127]. Finally, if  $S$  contains exactly one minimal left ideal and one minimal right ideal,  $K$  is the group of zero elements of  $S$  [Clifford and Miller, same vol., 117-125 (1948); these Rev. 9, 330].

D. Rees (Manchester).

## NUMBER THEORY

Kraitchik, Maurice. On the divisibility of factorials. Scripta Math. 14, 24-26 (1948).

Tables are given of the complete factorization of the functions  $n!-1$ ,  $n!+1$ ,  $P_n-1$  and  $P_n+1$  where  $P_n$  denotes the product of all primes not exceeding  $n$ . The upper limits for the four functions tabulated are  $n=21, 22, 47$  and  $53$ , respectively. Various small factors of other values of these functions are given also. The last function represents what

Schwarz, Š. Zur Theorie der Halbgruppen. Sborník Práce Prfrovedeckej Fakulty Slovenskej Univerzity v Bratislave no. 6, 64 pp. (1943). (Slovakian. German summary)

This paper deals with the structural theory of semigroups, their maximal subgroups and minimal ideals, and gives a definition of the radical of any semigroup having a kernel. Let  $\mathfrak{M}$  be a semigroup in which every element  $a$  has finite order, i.e.,  $a^{k+h}=a^k$  for some integers  $h, k \geq 1$ . To every idempotent element  $e$  of  $\mathfrak{M}$  there exists a unique maximal subgroup containing  $e$  as identity element. Two such groups belonging to different idempotents are disjoint. The semigroup  $\mathfrak{M}$  is the sum of its maximal subgroups if and only if each element  $a$  of  $\mathfrak{M}$  is such that  $a^{k+1}=a$  for some integer  $k \geq 1$ . Any minimal left (or right) ideal of  $\mathfrak{M}$  is a sum of isomorphic groups, a result proved for finite semigroups by Suschkewitsch [Math. Ann. 99, 30-50 (1928)]. The semigroup  $\mathfrak{M}$  is the sum of its minimal left ideals if and only if a relation  $a=xb$  ( $a, b, x \in \mathfrak{M}$ ) implies  $b=ya$  for some  $y \in \mathfrak{M}$ . Assume that the intersection  $\mathfrak{n}$  of all two-sided ideals of  $\mathfrak{M}$  (i.e. the Suschkewitsch kernel) is not vacuous;  $\mathfrak{n}$  plays a role analogous to the null ideal in ring theory. A left ideal  $\mathfrak{l}$  is called  $\mathfrak{n}$ -potent if  $\mathfrak{l}^\alpha \subseteq \mathfrak{n}$  for some positive integer  $\alpha$ . The radical  $\mathfrak{R}$  of  $\mathfrak{M}$  is defined to be the sum of all the  $\mathfrak{n}$ -potent two-sided ideals of  $\mathfrak{M}$ . If every element of  $\mathfrak{M}$  has finite order, and if  $\mathfrak{M}$  satisfies both the maximal and the minimal condition for left ideals, i.e., the ascending and descending chain conditions, and if  $\mathfrak{M}$  contains only one idempotent, then some power  $\mathfrak{M}^\alpha$  of  $\mathfrak{M}$  is a group (namely  $\mathfrak{n}$ ). If  $\mathfrak{M}$  is commutative, the same conclusion is reached without the condition that every element of  $\mathfrak{M}$  has finite order. A semigroup  $\mathfrak{M}$  is called simple if its only two-sided ideals are  $\mathfrak{M}$  and  $\mathfrak{n}$ . Any semigroup which is the sum of two disjoint groups is simple. A simple commutative semigroup is either itself a group, or the sum of two disjoint groups, or its square is a group.

A. H. Clifford (Baltimore, Md.).

Bates, Grace E. Decompositions of a loop into characteristic free summands. Bull. Amer. Math. Soc. 54, 566-574 (1948).

S'appuyant sur un travail antérieur [Amer. J. Math. 69, 499-550 (1947); ces Rev. 9, 8] l'auteur démontre que (comme dans un groupe) un composant d'une somme libre de loops ne peut être normal (c'est-à-dire être l'ensemble des éléments appliqués sur 0 par un homomorphisme). Elle prouve ensuite qu'un composant  $A$  d'une somme libre est caractéristique, c'est-à-dire conservé par tout automorphisme, s'il satisfait aux conditions nécessaires et suffisantes suivantes: (1)  $A$  n'a pas de composant de somme libre qui soit un loop libre, (2)  $A$  n'a pas de composant de somme libre qui soit isomorphe à un composant de  $B$ . (Le loop est supposé être la somme libre de  $A$  et de  $B$ .) Extension au cas d'un nombre quelconque de composants. Étude des décompositions en sommes libres dont les composants sont caractéristiques.

J. Kuntzmann (Grenoble).

are sometimes called Euclidean numbers since they occur in his proof of the infinity of primes. They are primes for  $n=2, 3, 5, 7, 11$  and  $31$ . D. H. Lehmer (Berkeley, Calif.).

Zirwes, Albert. Konstruktion und Klassifikation des Achtzells  $3^4$ . Pont. Acad. Sci. Acta 9, 103-126 (1945).

Since the  $n$ -dimensional "cube" has  $2^{n-1} \binom{n}{2}$   $g$ -dimensional elements, its total number of elements is  $\sum 2^{n-1} \binom{n}{2} = 3^n$ . This

may also be deduced from the fact that the center of a  $q$ -dimensional element of the cube  $(\pm 1, \pm 1, \dots)$  has coordinates consisting of  $q$  zeros and the rest  $\pm 1$ . These coordinate symbols, with  $(1, 1, \dots)$  added, represent the integers from 0 to  $3^n - 1$  in the ternary scale. By a suitable rearrangement of these numbers, the author constructs a "magic cube." The case when  $n=4$  is considered in detail. A series of sections by parallel hyperplanes is elegantly drawn on page 119. But the methods have no conspicuous advantage as compared with W. W. R. Ball [Mathematical Recreations and Essays, 11th ed., Macmillan, London, 1939; New York, 1947, pp. 217-220; these Rev. 8, 440] (where  $n=3$ , but the extension to  $n=4$  would present no difficulty).

H. S. M. Coxeter (Toronto, Ont.).

Pettineo, B. Sull'analisi indeterminata di primo grado. I. *Matematiche*, Catania 1, 33-37 (1945).

Betti's formula for the integral solutions of (1)  $c_1x_1 + \dots + c_m x_m = k$  has the defect of giving the same solution for an infinity of values of the parameters. The author gives a method of avoiding this defect. First it is proved that every integral solution of (1) for  $k=0$  has the form

$$x_i = a_i - a_1(c_1 a_1 + \dots + c_m a_m),$$

$i=1, \dots, m$ , where  $a_1, \dots, a_m$  are arbitrary parameters and  $a_1, \dots, a_m$  is a given integral solution of (1) for  $k=1$ . Let  $k$  be the smallest index for which  $a_k > 0$ ; if we take for  $a_k$  a member of a complete system of residues  $(\bmod a_k)$  then to every integral solution of (1) for  $k=0$  there exists one and only one system  $a_1, a_2, \dots, a_{k-1}, a_{k+1}, \dots, a_m$ . If  $k \neq 1$  in (1),  $ka_1, \dots, ka_m$  is a solution of (1) and hence all solutions of (1) may be found from

$$c_1(x_1 - ka_1) + \dots + c_m(x_m - ka_m) = 0.$$

N. G. W. H. Beeger (Amsterdam).

Pettineo, B. Sull'analisi indeterminata di primo grado. II. *Matematiche*, Catania 1, 38-41 (1945).

The number  $a_k$  [see the preceding review] is now eliminated. The equation is (1)  $c_1x_1 + \dots + c_m x_m = 0$ . Let  $\delta_m$  be the greatest common divisor of  $c_1, \dots, c_{m-1}$  and  $\gamma_i = c_i/\delta_m$ ,  $i=1, \dots, m-1$ ;  $(\delta_m, c_m)$  being 1,  $x_m = \delta_m y_m$  and (1) becomes (2)  $\gamma_1 x_1 + \dots + \gamma_{m-1} x_{m-1} + c_m y_m = 0$ . Let  $a_1, \dots, a_{m-1}$  be an integral solution of  $\gamma_1 x_1 + \dots + \gamma_{m-1} x_{m-1} = 1 - c_m$ ; then  $\gamma_1 a_1 + \dots + \gamma_{m-1} a_{m-1} + c_m \cdot 1 = 1$ . Hence  $a_1, \dots, a_{m-1}, 1$  is a solution of (2) (with second member 1). For  $a_k$  of the preceding review we can therefore take 1 and we can infer a solution of (1) with  $m-1$  parameters and  $a_k=0$  by the method of the preceding review. N. G. W. H. Beeger.

Pettineo, B. Sull'analisi indeterminata di grado superiore al primo. *Matematiche*, Catania 1, 42-48 (1945).

Parametric solutions of homogeneous Diophantine equations of degree greater than 2 are derived under special hypotheses; a method by Gloden [Mathesis 53, 233-235 (1939); these Rev. 1, 65] is generalized.

N. G. W. H. Beeger (Amsterdam).

Maggio, Oreste. Sistemi equitotali. *Matematiche*, Catania 1, 88-93 (1946).

Known proofs of known theorems: from

$$\sum_{i=1}^n x_i^k = \sum_{i=1}^n y_i^k,$$

$k=1, \dots, g$ , follows

$$\sum_{i=1}^n x_i^k + \sum_{i=1}^n (y_i + c)^k = \sum_{i=1}^n y_i^k + \sum_{i=1}^n (x_i + c)^k,$$

$k=1, \dots, g+1$ , for every  $c$ ;  $g < n$ , otherwise the  $y_i$  are, in a certain order, identical with the  $x_i$  [cf. Dorwart, Bull. Amer. Math. Soc. 53, 381-391 (1947); these Rev. 8, 442, and the references given in that review].

N. G. W. H. Beeger (Amsterdam).

Novikov, A. P. A new solution of the indeterminate equation  $ax^3 + by^3 + cz^3 = 0$ . *Doklady Akad. Nauk SSSR (N.S.)* 61, 205-206 (1948). (Russian)

The author considers the Diophantine equation

$$(1) \quad ax^3 + by^3 + cz^3 = 0,$$

$a, b, c$ , integers, to be solved in integers  $x, y, z$ , which are relatively prime by pairs. He proves that if (2)  $x=1, y=\beta, z=\gamma$ , is a solution of (1), then all solutions are given by

$$(3) \quad \begin{cases} \pm x = t_1 + t_2, \\ \pm y = \beta(t_1 - t_2) - 2t_1\gamma c, \\ \pm z = \gamma(t_1 - t_2) + 2t_1\beta b, \end{cases}$$

where  $t_1, t_2, t_3$  are parameters taking on integral values which satisfy (4)  $t_1 t_2 = t_3^3 bc$ . From the fact that (2) satisfies (1) we have the condition (5)  $a = -(b\beta^3 + c\gamma^3)$ . Using (4) and (5) one readily verifies that the  $x, y, z$ , given in (3), satisfy (1). Conversely, starting with an arbitrary solution  $x, y, z$  of (1), consider the equations

$$\begin{cases} \pm 2ax = t_1 + t_2, \\ \pm 2ay = \beta(t_1 - t_2) - 2t_1\gamma c, \\ \pm 2az = \gamma(t_1 - t_2) + 2t_1\beta b, \end{cases}$$

and solve for  $t_1, t_2, t_3$ , as

$$\begin{cases} t_1 = ax \pm (\beta by + \gamma cz), \\ t_2 = ax \mp (\beta by + \gamma cz), \\ t_3 = \beta z - \gamma y. \end{cases}$$

Then using (5) one can show that these  $t_i, i=1, 2, 3$ , satisfy (4). Replacing  $t_i$  by  $2at_i$ , and noting that (4) still holds, completes the proof. H. N. Shapiro (New York, N. Y.).

Hua, Loo-keng, and Min, Szu-ho. On a double exponential sum. *Sci. Rep. Nat. Tsing Hua Univ.* 4, 484-518 (1947).

A detailed proof is given of the following theorem, announced earlier [Acad. Sinica Science Record 1, 23-25 (1942); these Rev. 5, 255]. Let  $K$  be a finite field with  $p^n$  elements, suppose  $f(x, y)$  is a polynomial of degree  $n \geq 4$  with coefficients in  $K$  which cannot be expressed as a polynomial in a single variable (which is itself a linear function of  $x$  and  $y$ ), and let  $S[a]$  denote the trace of the element  $a$  in  $K$  (with respect to the included prime field); then for large  $p$  we have

$$(4) \quad \sum_{x, y \in K} \exp \{2\pi i S[f(x, y)]/p\} = O(p^{(2-3/n)}),$$

where  $x$  and  $y$  run independently over all elements in  $K$  and the constant implied by  $O$  depends only on  $n$ .

If we write  $f(\alpha_1 t + \beta_1, \alpha_2 t + \beta_2) = \sum_{r=0}^n F_r(\alpha_1, \alpha_2, \beta_1, \beta_2) t^r$  and denote by  $M$  the matrix of  $n$  rows and 4 columns whose  $i$ th row is  $\partial F_i / \partial \alpha_1, \partial F_i / \partial \alpha_2, \partial F_i / \partial \beta_1, \partial F_i / \partial \beta_2$ , then the proof divides itself into two cases according to whether the rank of  $M$  is 4 or less. In the former case the proof is based upon two lemmas concerning the solution of polynomial equations and upon the one-variable analogue of the desired theorem, namely:  $\sum_{x \in K} \exp \{2\pi i S[f(x)]/p\} = O(p^{(1-1/n)})$ .

where  $f(x)$  is a polynomial of  $n$ th degree with coefficients in  $K$ . This case of the proof (rank of  $M$  equal to twice the number of variables) can be easily generalized to any number of variables and this has been published by Min [Quart. J. Math., Oxford Ser. 18, 133–142 (1947); these Rev. 9, 175].

The case in which the rank of  $M$  is less than four (twice the number of variables) is much more involved and seems very difficult to generalize to more variables. The crucial lemma is to the effect that in this case  $f(x, y)$  can be transformed by a nonsingular linear transformation in  $K$  to a polynomial  $g(\xi, \eta)$  which is either of degree at most  $n/2$  in  $\eta$  or a polynomial in  $\xi^2 + \eta^2 (nK)$ . Several rather specialized lemmas concerning polynomials in a finite field are needed to get this.

In an appendix the authors show that stronger results than (•) are true for  $n=2, 3$ . For  $n=2$  an explicit evaluation can be given, the absolute value of which is exactly  $p^m$  (for  $p>2$ ). By making use of this fact the authors get  $O\{p^{m(2-b)}\}$  instead of  $O\{p^{m(2-b)}\}$  for  $n=3$ . In the paper referred to above Min says that he has obtained improvements of (•) for other values of  $n$  analogous to those obtained by Davenport in the case of one variable (with  $m=1$ ) [J. Reine Angew. Math. 169, 158–176 (1933)].

P. T. Bateman (Princeton, N. J.).

Hua, L. K., and Vandiver, H. S. On the existence of solutions of certain equations in a finite field. Proc. Nat. Acad. Sci. U. S. A. 34, 258–263 (1948).

The authors consider the number  $N(p^n; c_1, \dots, c_{s+1})$  of solutions in nonzero  $x$ 's of the equation  $c_1x_1^n + c_2x_2^n + \dots + c_sx_s^n + c_{s+1} = 0$ , where  $s$  is a fixed integer not less than 2, the coefficients and unknowns are elements in the finite field  $GF(p^n)$ , the exponents  $c_k$  are fixed (distinct) positive integers,  $c_1c_2 \dots c_s \neq 0$ , and  $c_{s+1} \neq 0$  in the case  $s=2$ . They prove in two ways that the minimum of the  $N(p^n; c_1, \dots, c_{s+1})$  (for all permissible choices of the  $c_k$ ) tends to infinity with  $p^n$ . More specifically the following estimates are obtained:

$$|N(p^n; c_1, c_2, c_3) - p^n| < (a_1a_2)^2 p^{n/2}, \quad s=2, c_3 \neq 0, \\ |N(p^n; c_1, \dots, c_{s+1}) - (p^n - 1)/p^n| < (a_1 \dots a_s)p^{n/2}, \quad s > 2.$$

The first approach is based on Vandiver's previous reduction of the problem to the case  $s=2$  [same Proc. 32, 47–52 (1946); these Rev. 7, 365] and his consideration of that case [same Proc. 33, 236–242 (1947); these Rev. 9, 9]. The second method (which is subject to the limitation  $s>2$ ) is somewhat more direct; it is similar to the method used by Mordell in the study of generalized Gaussian sums [Quart. J. Math., Oxford Ser. 3, 161–167 (1932); cf. also the preceding review]. P. T. Bateman (Princeton, N. J.).

Vandiver, H. S. Congruence methods as applied to Diophantine analysis. Math. Mag. 21, 185–192 (1948).

[The author's initials are misprinted H. V. in the original.] This paper contains a simple account of the author's method of constructing Diophantine equations that have no solutions, by making sure that the corresponding congruences have no solution for a properly chosen prime modulus [see, for example, Proc. Nat. Acad. Sci. U. S. A. 32, 101–106 (1946); these Rev. 7, 414]. Thus it is shown how to find Diophantine equations of the form  $Au^n + Bv^n = 1$  that have no solutions; the modulus used is a prime of the form  $mc+1$ . An example is the equation  $7u^4 - 10v^4 = 1$ , where the modulus is the prime 13. A general method is also given whereby impossible Diophantine equations are obtained by

a certain process of addition from other impossible equations and congruences.

H. W. Brinkmann.

Huff, Gerald B. Diophantine problems in geometry and elliptic ternary forms. Duke Math. J. 15, 443–453 (1948).

The author proves first the following extension of a result by G. Sansone [Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 2, 124–128 (1941); these Rev. 8, 220]: if  $e_1, e_2, e_3$  are distinct numbers in some field  $k$  of algebraic numbers, then a point  $(x, y)$  in  $k$  on the curve  $y^2 = (x+e_1)(x+e_2)(x+e_3)$  is the tangential of a point in  $k$  on the same curve if, and only if,  $x+e_1, x+e_2, x+e_3$  are all squares of numbers in  $k$ . This enables him to go deeply into a procedure of finding new rational points on an elliptic cubic, as tangent points of given rational points, leading to a geometric characterization of the plane cubic  $C: az_1(z_2^2 - z_3^2) = bz_2(z_1^2 - z_3^2)$ , which is then investigated for different fields. Thus, as a consequence of more general results, it is shown that (when  $k$  is the field of rational numbers) the curve  $C$  contains either eight rational points or an infinite number of rational points if  $a = l^2 + m^2, b = lm$ , where  $l$  and  $m$  are relatively prime rational integers; moreover,  $C$  contains a set of sixteen rational points (never extendible either by the tangents and chords process or by the procedure above) if  $a, b$  and  $a+b$  are all squares of rational integers which are relatively prime each to each. Applications are made to three geometric problems, one of them giving a new infinite planar rational distance set, i.e., a set of rational points lying on a plane and having rational distances two by two.

B. Segre (Bologna).

Brauer, Alfred. On the irreducibility of polynomials with large third coefficient. Amer. J. Math. 70, 423–432 (1948).

Let  $f(x) = x^n + a_1x^{n-1} + \dots + a_n$  ( $a_n \neq 0$ ) be a polynomial with integral coefficients. Perron [J. Reine Angew. Math. 132, 288–307 (1907)] and later Lipka [Math. Naturwiss. Anz. Ungar. Akad. Wiss. 54, 349–357 (1936)] showed that  $f(x)$  is irreducible in the field  $P$  of rational numbers if one of the numbers  $|a_1|$  or  $|a_2|$  is sufficiently large compared with certain functions of the other coefficients. The author obtains more general results in the same direction. His main result is the following. Let  $m$  be the minimum of the partial sums of the series  $0 + a_3 + a_4 + \dots + a_n$  and  $m^*$  the minimum of the partial sums of  $0 - a_3 + a_4 - \dots + (-1)^n a_n$ . Set  $t = |1 + a_1| + |a_1 + a_2| + |a_2| + \dots + |a_n|$ . If  $a_2 > \max(t, \frac{1}{2}a_1^2 + |m^*|)$  for  $a_1 \geq 0$ ,  $a_2 > \max(t, \frac{1}{2}a_1^2 + |m|)$  for  $a_1 \leq 0$ , then  $f(x)$  is irreducible in  $P$ .

T. Nagell (Uppsala).

Bambah, R. P., and Chowla, S. On the sign of the Gaussian sum. Proc. Nat. Inst. Sci. India 13, 175–176 (1947).

The sign of the Gaussian sum  $S = \sum_{n=0}^{k-1} e^{2\pi i n^2/k}$  ( $k$  an odd integer greater than 1) is determined by making use of a result, due to van der Corput, whereby it is possible to estimate the difference between a sum of the form  $\sum_n e^{2\pi i f(n)}$  and the corresponding integral  $\int_a^b e^{2\pi i f(x)} dx$  [Math. Ann. 84, 53–79 (1921)]. It is thus possible to make an estimate of  $S$  by means of the corresponding integral, which is in turn estimated by the second mean value theorem. As a result the sign of  $S$  is readily determined for  $k > 49$ . The idea of this determination of the sign of  $S$  is somewhat similar to, but simpler than, the one used by van der Corput himself, in the paper referred to above.

H. W. Brinkmann.

**Amante, Salvatore.** Sulle dedotte e controdedotte numeriche di una funzione numerica. *Matematiche*, Catania 2, 4-9 (1947).

Following Cipolla, the quotient  $f(n)$  of two numerical functions  $g(n)$  and  $h(n)$  is defined by the relation  $f(g(n)) = h(n)$ . The author determines the form of  $f(n)$  in two cases where  $g(n)$  and  $h(n)$  are given.

R. Bellman.

**Gage, Walter H.** Proof of a formula of Liouville. *Bull. Amer. Math. Soc.* 54, 581-586 (1948).

The author asserts that this is the first elementary proof of formula (Q) of Liouville's set of general formulas in the theory of numbers. It is accomplished by the ingenious use of the usual methods and employs only elementary operations. However, at one point it involves the formal manipulation of a series which appears to be infinite and perhaps not even convergent. H. S. Zuckerman (Seattle, Wash.).

**Petr, K.** On alternating functions in a cyclotomic field. *Rozpravy II. Třídy České Akad.* 56, no. 3, 12 pp. (1946). (Czech)

Gause proved [Werke, v. 2, pp. 155-158] that

$$(1) \quad \sum_{k=1}^{P-1} \left( \frac{k}{P} \right) \alpha^k = \psi_P(\alpha),$$

where  $P$  is an odd squarefree number,  $\alpha$  a primitive root of  $x^P = 1$ , and

$$\psi_P(\alpha) = (\alpha - \alpha^{-1})(\alpha^3 - \alpha^{-3}) \cdots (\alpha^{P-3} - \alpha^{-P+3}).$$

The author calls attention to Cauchy's little known proof of (1) [J. Math. Pures Appl. 5, 154-168 (1840)] for odd primes  $P$  and shows how this proof can be extended to the general case of any odd squarefree  $P$  by means of the identity  $\psi_{P_1}(\alpha_1) \cdot \psi_{P_2}(\alpha_2) = \psi_P(\alpha)$ , where  $P_1, P_2$  are odd, squarefree and relatively prime,  $\alpha_i$  primitive roots of  $x^{P_i} = 1$ ,  $P = P_1 P_2$  and  $\alpha = \alpha_1 \alpha_2$ . The proof is purely algebraic (e.g., does not use the representation of  $\alpha - \alpha^{-1}$  as  $2i \sin 2\pi/P$ ) and rests mainly on the law of reciprocity for the Legendre-

Jacobi symbol  $\left( \frac{P}{Q} \right)$ . A simple proof of this law based on the alternating function

$$\varphi(\alpha) = \prod_{k=1}^{1(P-1)} (\alpha^k - \alpha^{-k})$$

is offered. By definition of  $\left( \frac{Q}{P} \right)$ ,  $\varphi(\alpha^Q) = \left( \frac{Q}{P} \right) \varphi(\alpha)$ . If  $\alpha, \beta$  are primitive roots of  $x^P = 1, x^Q = 1$ , and if

$$W = \prod_{j=1, 2, \dots, \frac{1}{2}(P-1)} (\alpha^j \beta^k - \alpha^{-j} \beta^{-k}) \quad k=1, 2, \dots, Q-1,$$

then  $W\varphi(\alpha) = \varphi(\alpha^Q)$ , whence  $W = \left( \frac{Q}{P} \right)$ . Similarly,

$$W' = \prod_{j=1, 2, \dots, P-1} (\alpha^j \beta^k - \alpha^{-j} \beta^{-k}) \quad k=1, 2, \dots, \frac{1}{2}(Q-1),$$

is equal to  $\left( \frac{P}{Q} \right)$ . But  $W'$  is obtained from  $W$  by changing the signs of  $\frac{1}{2}(P-1)(Q-1)$  factors. F. A. Behrend.

**Skolem, Th.** A property of ternary quadratic forms and its connection with the quadratic reciprocity theorem. *Norsk Mat. Tidsskr.* 30, 1-10 (1948). (Norwegian)

Let  $F$  be a homogeneous form with integral coefficients in the variables  $x, y, z, \dots$ . The prime  $p$  is said to be a divisor of  $F$  if the congruence  $F \equiv 0 \pmod{p^a}$  is solvable for

all positive integers  $n$  in integral values of the variables not all divisible by  $p$ . Any other prime is a nondivisor of  $F$ . When  $F=0$  is solvable in integers (not all zero) the symbolic "infinite" prime  $p = p_\infty$  is said to be a divisor of the form. The following theorem holds. The number of the nondivisors of a ternary quadratic form is an even number. The author gives a new proof of this "nondivisor theorem" based upon the quadratic reciprocity theorem. Inversely he shows that the latter theorem can be derived from the following special cases of the nondivisor theorem: (1) if no odd nondivisor exists, the primes 2 and  $p_\infty$  are both divisors or they are both nondivisors; (2) if exactly one odd nondivisor exists, then either 2 is a divisor and  $p_\infty$  a nondivisor or inversely. This proof of the quadratic reciprocity theorem is in reality (expressed in another language) the same as the second demonstration of Gauss [Werke, v. 1, pp. 284-286].

T. Nagell (Uppsala).

**Sominsky, I.** Sur les limites du domaine fondamental d'un groupe d'automorphismes d'une forme ternaire quadratique indéfinie. *C. R. (Doklady) Acad. Sci. URSS (N.S.)* 56, 127-128 (1947).

Let  $f(x, y, z)$  be a ternary indefinite quadratic form with integer coefficients of positive determinant and not representing zero. Let  $\mathfrak{G}$  be the group of automorphisms of  $f(x, y, z)$ ,  $\Omega$  be a fundamental domain and  $\gamma_1, \dots, \gamma_k$  the boundaries of  $\Omega$ , which separate  $\Omega$  from its neighboring fundamental domains  $\Omega_1, \dots, \Omega_k$ . In a preceding paper [Leningrad State Univ. Annals [Uchenye Zapiski] Math. Ser. 10, 148-153 (1940); these Rev. 2, 252] the author proved that  $\gamma_i$  is in general a quadratic surface. In this note he establishes that  $\Omega$  can be deformed into another fundamental domain  $\bar{\Omega}$  whose boundaries are planes.

L.-K. Hua (Urbana, Ill.).

**Sominskii, I. S.** On the structure of the group of automorphisms of a ternary quadratic indefinite form. *Doklady Akad. Nauk SSSR (N.S.)* 56, 241-243 (1947). (Russian)

Let  $f(x, y, z) = ax^2 + by^2 + cz^2 + 2gyz + 2hxz + 2kxy$  be a ternary indefinite quadratic form with integer coefficients  $a, b, c, g, h, k$  of positive determinant and not representing zero. Let  $\mathfrak{G}$  be the group of automorphisms of  $f$ , let  $\Omega$  be a fundamental region of  $\mathfrak{G}$  and let  $\gamma_1, \dots, \gamma_k$  be the boundaries of  $\Omega$ , which separate  $\Omega$  from its neighboring fundamental domains  $\Omega_1, \dots, \Omega_k$ . Let  $S_1, \dots, S_k$  be the transformations carrying  $\Omega$  into  $\Omega_1, \dots, \Omega_k$ . The author proves that  $S$  is either of order two or of order infinity. The method based on the fact that the order of a modular transformation of finite order can only be 2, 3, 4 or 6. Then he works out the cases admitting automorphisms of order 3, 4 and 6. L.-K. Hua (Urbana, Ill.).

**Inkeri, K.** Über den Euklidischen Algorithmus in quadratischen Zahlkörpern. *Ann. Acad. Sci. Fenniae. Ser. A. I. Math.-Phys.* no. 41, 35 pp. (1947).

Let  $m$  be a square-free integer. The quadratic field  $K(\sqrt{m})$  is called Euclidean if, for any pair of integers  $\alpha$  and  $\mu (\neq 0)$  in  $K(\sqrt{m})$ , there is an integer  $\beta$  so that  $|N(\alpha - \mu\beta)| < |N(\mu)|$ , where  $N(\mu)$  denotes the norm of  $\mu$  in  $K(\sqrt{m})$ . After the contributions of more than a dozen authors, the conclusion arrived at is that  $K(\sqrt{m})$  is Euclidean if  $m = -11, -7, -3, -2, -1, 2, 3, 5, 6, 7, 11, 13, 17, 19, 21, 29, 33, 37, 41, 57, 73, 97$  (22 cases) and that the cases with  $m$  a prime  $p \equiv 1 \pmod{24}$  and  $p \leq e^{30}$  remain in

doubt. The author proves that, for  $97 < p < 5000$ ,  $K(\sqrt{m})$  is not Euclidean. Practically, the most essential part of the paper is the proof of the cases  $p = 193, 241, 313, 337, 457, 601$ .

As Davenport [abstract in Bull. Amer. Math. Soc. 54, 662 (1948)] reduced the reviewer's upper bound  $p \leq e^{\frac{1}{2}}$  [Hua, Trans. Amer. Math. Soc. 56, 537-546 (1944); these Rev. 6, 170] to  $p \leq 2^4$ , the only doubtful range is  $5000 < p < 2^4$ . It has been verified that  $K(\sqrt{p})$  is not Euclidean except possibly in the six cases mentioned before [Min and the reviewer verified it up to  $p \leq 10^4$ , and a forthcoming paper of Chatland [abstract in Bull. Amer. Math. Soc. 54, 828 (1948)] contains the verification up to  $p \leq 2^4$ ]. Therefore we can conclude now that there are precisely 22 quadratic Euclidean fields.

The paper also contains some simplified treatments of previous results.

L.-K. Hua (Urbana, Ill.).

Krasner, Marc. Certaines propriétés des séries de Taylor d'un ensemble au plus dénombrable de variables dans les corps valués complets et une démonstration structurale des formules de M. Pollaczek. Bull. Sci. Math. (2) 71, 123-152, 180-200 (1947).

For the formulas of Pollaczek [same Bull. (2) 70, 199-218 (1946); these Rev. 9, 273] see (4) below. The author develops at some length a theory of Taylor series with coefficients and variables in an ultrametric field  $K$ , that is, a field which is closed with respect to a non-Archimedean valuation  $|\alpha|$ ,  $\alpha \in K$ ,  $|\alpha + \beta| \leq \max\{|\alpha|, |\beta|\}$ . He considers derangement, substitution, uniqueness, inversion, sequences of series, continuity and differentiability, as far as these are needed for his primary purpose. Since a series  $\sum \alpha_n$  converges in  $K$  if and only if  $|\alpha_n| \rightarrow 0$  as  $n \rightarrow \infty$ , derangement, for example, is unrestricted, and the theory also differs in other ways from that for the complex field. The notion of majorant plays an important rôle in the theory as developed, where  $\sum A_i$  is said to majorize  $\sum \alpha_i$  if  $|\alpha_i| \leq |A_i|$ , for all  $n$ , and a number  $A$  of  $K$  majorizes  $\sum \alpha_n$  if  $|\alpha_n| \leq |A|$ , for all  $n$ . Henceforth assume further that  $K$  has characteristic zero, that its residue class field has characteristic  $p \neq 0$ , and that it is algebraically closed. The series  $\log(1+x) = x - \frac{1}{2}x^2 + \dots$ ,  $e^x = 1 + x/1! + x^2/2! + \dots$  converge in  $K$  if and only if  $|x| < 1$ , and  $|x| < \lambda = |p^{1/(p-1)}|$ , respectively, and define functions having the usual properties with the restrictions on  $x \in K$ . The author applies his lemmas on series to prove the following:

$$(1) \quad l_j(\alpha) = \left[ \frac{\partial^j \log [1 + \sum_{i=0}^{\infty} \alpha_i (e^x - 1)^i]}{\partial x^j} \right]_{x=0}, \quad j = 1, 2, \dots,$$

is a convergent series in the denumerably infinite set of variables  $(\alpha)$ , at least inside the parallelopiped  $|\alpha_i| < \lambda^{-i}$ ,  $i = 0, 1, \dots$ ; this extends the definition of the Kummer logarithmic derivatives;

$$(2) \quad L_j(\alpha) = \lim_{n \rightarrow \infty} l_{jn}(\alpha), \quad j = 1, 2, \dots,$$

is a convergent series in  $(\alpha)$  for  $|\alpha_i| < d^{-i}$ ,  $i = 0, 1, \dots$ , for any positive  $d > \lambda$ , and  $L_j(\alpha) = L_{j_1}(\alpha)$ , if  $j_1 = j_2 \pmod{p-1}$ ; the  $L_j$  are called the logarithmic limits of Pollaczek;

$$(3) \quad \log_m(\alpha) = \log \left( 1 + \sum_{i=0}^{\infty} \alpha_i (\xi^m - 1)^i \right), \quad m = 0, 1, \dots, p-1,$$

where  $\xi$  is a primitive  $p$ th root of unity in  $K$ , is a convergent series in  $(\alpha)$  for  $|\alpha_i| < d^{-i}$ ,  $d > \lambda$ ; these are the so-called

$p$ -adic logarithms;

$$(4) \quad \begin{aligned} L_0(\alpha) + \sum_{m=0}^{p-1} (\log_m(\alpha) - \log_m(\alpha)), \\ L_j(\alpha) = \frac{(-1)^j}{p} \left( \sum_{m=1}^{p-1} \xi_m^{-j} \right) \sum_{m=1}^{p-1} \xi_m^{-j} \log_m(\alpha), \end{aligned} \quad j = 1, \dots, p-1,$$

where  $\xi_m = (\lim_{n \rightarrow \infty} m^p)$  is a  $(p-1)$ th root of unity in  $K$ . In the proof of (4), use is made of the property of  $K$  that it contains subfields isomorphic to the  $p$ -adic extension  $k_p$  of the rational field  $k$  and to  $k_p(\xi)$ , respectively. If  $\alpha_0, \alpha_1, \dots$  are in  $k_p$ , so are the  $L_j$ , while  $\log_m(\alpha)$  is in  $k_p(\xi)$ . Then (4) is valid in  $k_p(\xi)$ , and hence includes Pollaczek's relations. [The section of the paper beginning on p. 180 apparently should be numbered 4, not 9.] R. Hull (Lafayette, Ind.).

Cohen, Eckford. Sums of an odd number of squares in  $GF[p^n, x]$ . Duke Math. J. 15, 501-511 (1948).

The author finds a formula for the number of solutions of  $F = a_1 X_1^2 + \dots + a_{2k+1} X_{2k+1}^2$ ,  $\deg X_i < k$ , where  $a_1, \dots, a_{2k+1}$  are nonzero elements of  $GF(p^n)$ ,  $p > 2$ , and  $F$  is a given polynomial in  $GF[p^n, x]$ , of degree less than  $2k$ . The formula involves the Artin  $\sigma$ -function [Math. Z. 19, 207-246 (1924)]:  $\sigma_i(F) = \sum_{\deg H=i} (F/H)$ , where  $(F/H)$  indicates the quadratic character of  $F$  with respect to  $H$ , and the sum is over primary  $HeGF[p^n, x]$ , of degree  $i$ . The corresponding problem for an even number of squares was treated earlier by the author [same J. 14, 251-267, 543-557 (1947); these Rev. 9, 81, 176]. As in the even case, in which the formulas involved certain divisor functions, the present formulas also reduce to those of Jordan and Dickson when  $k = 1$ ,  $HeGF(p^n)$ , and they become much simpler for other special restrictions on  $F$ .

R. Hull (Lafayette, Ind.).

Alder, Henry L. The nonexistence of certain identities in the theory of partitions and compositions. Bull. Amer. Math. Soc. 54, 712-722 (1948).

Let  $q_{d,m}(n)$  denote the number of partitions of  $n$  into parts differing by at least  $d$ , each part being greater than or equal to  $m$ . The following theorems are proved. (1) Unless  $d = 1$  or  $d = 2$ ,  $m = 1, 2$ ,  $q_{d,m}(n)$  is not equal to the number of partitions of  $n$  into parts taken from any set of integers. (2) Unless  $d = 1$ ,  $q_{d,m}(n)$  is not equal to the number of partitions of  $n$  into distinct parts taken from any set of integers. (3) The number of partitions of  $n$  into parts differing by at least  $d$  and where parts divisible by  $d$  differ by at least  $2d$  is not equal to the number of partitions of  $n$  into parts taken from any set of integers if  $d > 3$ . (4) The number of compositions of  $n$  into parts differing by at least  $d$ , each part being greater than or equal to  $m$ , is not equal to the number of compositions of  $n$  into parts taken from any set of integers.

T. Estermann (London).

de Bruijn, N. G. On Mahler's partition problem. Nederl. Akad. Wetensch., Proc. 51, 659-669 = Indagationes Math. 10, 210-220 (1948).

By considering the functional equation

$$(f(z+\omega) - f(z))/\omega = f(qz)$$

K. Mahler [J. London Math. Soc. 15, 115-123 (1940); these Rev. 2, 133] obtained an approximate formula for the number  $p(h)$  of solutions of  $h = h_0 + h_1 r + h_2 r^2 + \dots$  in non-negative integers  $h_0, h_1, h_2, \dots$ , where  $r \geq 2$  is a given integer. The author investigates this problem by a different method and sharpens Mahler's result. He proves, in particular, that,

for  $h \rightarrow \infty$ ,

$$(1) \quad \log p(rh) = \frac{1}{2 \log r} \left( \log \frac{h}{\log h} \right)^2 + \left( \frac{1}{2} + \frac{1}{\log r} + \frac{\log \log r}{\log r} \right) \log h - \left( 1 + \frac{\log \log r}{\log r} \right) \log \log h + \psi \left( \frac{\log h - \log \log h}{\log r} \right) + o(1),$$

where  $\psi$  is a periodic function with period 1.

The outline of the argument is as follows. For any real  $u$  let  $P(u)$  denote the number of solutions of  $h_0 + h_1 r + h_2 r^2 + \dots \leq u$  in nonnegative integers  $h_0, h_1, h_2, \dots$ , and write  $P_1(u) = \int_{u-1}^u P(v) dv$ . We introduce the generating function

$$(2) \quad F(s) = \prod_{k=0}^{\infty} (1 - e^{-rs})^{-1} = \int_{-\infty}^{\infty} e^{-su} dP(u), \quad \Re s > 0,$$

and prove without difficulty that

$$(3) \quad P_1(u) = r(2\pi i)^{-1} \int_{s-i\infty}^{s+i\infty} s^{-2} F(s) e^{us/r} ds, \quad a > 0.$$

Next, by Mellin's inversion formula and (2), we obtain rather accurate information about the behaviour of  $F(s)$  near  $s=1$ ; this enables us to evaluate approximately the integral (3), and so leads to an asymptotic formula for  $\log P_1(u)$ . A corresponding result for  $P(u)$  is now easily deduced, and finally (1) follows by virtue of the relation  $p(rh) = P(h)$ .  
L. Mirsky (Sheffield).

Zahlen, Jean-Pierre. *Sur les nombres premiers à une suite d'entiers consécutifs*. Euclides, Madrid 8, 115-121 (1948).

Pillai [Proc. Indian Acad. Sci., Sect. A 11, 6-12 (1940); these Rev. 1, 199] has proved that in every set of less than 17 consecutive integers there is at least one integer which is relatively prime to all the others, and that there are sequences of  $k$  integers,  $17 \leq k \leq 430$ , which do not have this property. A. Brauer [Bull. Amer. Math. Soc. 47, 328-331 (1941); these Rev. 2, 248] has shown that the second part of the result of Pillai holds for all  $k \geq 17$ . In the present paper, the author determines conditions which, when imposed on a sequence of  $k$  consecutive integers, ensure that at least one integer is relatively prime to all the others. The following are examples of the results, which are obtained by elementary means. (1) If a set of  $k$  consecutive integers contains a prime, it contains at least one integer relatively prime to all the others. (2) Among six consecutive sets of 17 consecutive integers, there are at least five sets which contain an integer relatively prime to all the others in the set.

R. D. James (Vancouver, B. C.).

Ramaswami, V. On the number of integers  $\leq X$  and free of prime divisors  $> X^{\epsilon}$ . Science and Culture 13, 503 (1948).

Ramaswami, V. On the number of integers  $\leq X$  and free of prime divisors  $> X^{\epsilon}$ , and a problem of S. S. Pillai. Science and Culture 13, 503 (1948).

Ramaswami, V. On the number of positive integers  $< X$  and free of prime divisors  $> X^{\epsilon}$ . Science and Culture 13, 465 (1948).

In letters to the editor, the author gives asymptotic formulas for  $f(x, \epsilon)$ , the number of positive integers less

than  $x$  and free of prime divisors greater than  $x^{\epsilon}$ . Proofs will appear elsewhere. R. D. James (Vancouver, B. C.).

Segal, B. L. Trigonometric sums and some of their applications in the theory of numbers. *Uspehi Matem. Nauk (N.S.)* 1, no. 3-4 (13-14), 147-193 (1946). (Russian)

This is a connected account of the estimation of trigonometric sums with applications to the distribution of primitive roots, indices and residues of powers, and to Waring's problem. The author's aim is to provide an introduction to the analytical theory of numbers for a wide circle of readers, rather than to carry the specialist to the frontiers of knowledge. He accordingly regards Vinogradov's methods as lying beyond his scope. He follows Hardy and Littlewood in basing his treatment of Waring's problem on Weyl's estimations, but he also includes an account of the complement supplied by Hua in 1938, so that his ultimate objective is the inequality  $G(k) \leq 2^k + 1$ , with the corresponding asymptotic formula. The exposition is full and clear, and the reader with a moderate knowledge of analysis and arithmetic is led by easy stages from the most elementary applications to the intricacies of Waring's problem.

A. E. Ingham (Cambridge, England).

Anfert'eva, E. A. On the transformation formulas of Vinogradov-Corput. *Doklady Akad. Nauk SSSR (N.S.)* 60, 541-544 (1948). (Russian)

In this paper the author obtains the following approximate transformation formulas:

$$(I) \quad S(x) = \sum_{n=1}^{\infty} e^{-n^m/N} e^{-2\pi i \alpha n^m} = A |\alpha|^{-1/2(m-1)} \sum_{n=1}^{\infty} \frac{1}{n^{1-1/2(m-1)}} \times \exp \{-n^{m/(m-1)} |\alpha|^{-1/(m-1)} / (2\pi N \alpha (m-1))\} \times \exp \{i n^{m/(m-1)} |\alpha|^{-1/(m-1)}\} + O(\ln^2 |\alpha| N),$$

$$(II) \quad T(x) = \sum_{n=1}^{\infty} \tau(n) e^{-n^m/N} e^{-2\pi i \alpha n^m} = B |\alpha|^{-1/(2m-1)} \sum_{n=1}^{\infty} \frac{\tau(n)}{n^{1-1/2(2m-1)}} \times \exp \{-n^{m/(2m-1)} |\alpha|^{-1/(2m-1)} / (2\pi N \alpha (m-1))\} \times \exp \{-i n^{m/(2m-1)} |\alpha|^{-1/(2m-1)}\} + O(\ln^2 |\alpha| N),$$

where  $x = (N^{-1} + 2\pi i \alpha)^{1/m}$ ,  $m \geq 2$ ,  $2\pi |\alpha| N > 1$ ,  $A, B$  are constants, and  $\tau(n)$  denotes the number of divisors of  $n$ . The proof of (I) is sketched and consists roughly of first representing  $S(x)$  as

$$S(x) = \frac{1}{2\pi m i} \int_{s-i\infty}^{s+i\infty} x^{-s} \Gamma(s/m) \zeta(s) ds,$$

then shifting the path of integration to the line  $s=0$  and expressing  $\zeta(s)$  in terms of  $\zeta(1-s)$  by using the well-known functional equation for the Riemann zeta function  $\zeta(s)$ . This, together with standard estimates for  $\Gamma(1-s)$ ,  $\Gamma(s/m)$ , and  $\sin(\frac{1}{2}\pi s)$ , leads to (I). It is asserted then that (II) follows in an analogous way if  $\zeta^2(s)$  is used instead of  $\zeta(s)$ .

H. N. Shapiro (New York, N. Y.).

\*Ammann, André. *Quelques Propriétés Concernant la Répartition des Suites de Nombres Modulo Un*. Thesis, University of Geneva, 1947. 39 pp.

Soit  $\bar{x}$  le reste mod  $(\omega)$  de  $x$ . On sait qu'une suite réelle  $\{x_i\}$  est dite équirépartie mod  $(\omega)$  si,  $f_n$  étant le rapport à

$n$  du nombre de ceux des  $n$  premiers  $x_i$  qui appartiennent à une partie  $(\alpha, \beta)$  de  $(0, \omega)$ , on a:  $\lim_{n \rightarrow \infty} f_n = (\beta - \alpha)/\omega$ , et cela que soient  $\alpha$  et  $\beta$ ; ou encore (définition équivalente), si l'on a pour toute fonction  $P(x)$  de période  $\omega$  et intégrable- $R$ ,  $\lim_{n \rightarrow \infty} F_n = [P]$ , où  $[P] = \omega^{-1} \int_0^\omega P(x) dx$  et  $F_n = \sum_{i=1}^n P(x_i)/n$ . L'auteur introduit les définitions nouvelles suivantes: (a)  $\{x_i\}$  est dite "unifiante" mod  $(\omega)$  si l'on a la propriété plus faible: (1)  $\liminf F_n \leq [P] \leq \limsup F_n$ , qui entraîne, si  $P(x)$  est bornée, qu'un des points d'accumulation de la suite  $\{F_n\}$  est égal à  $[P]$  [l'unifiante est réalisée si seulement (1) est satisfaite pour les  $P(x)$  de la forme  $a_0 + \sum_{k=1}^l (a_k \cos 2k\pi\omega^{-1}x + b_k \sin 2k\pi\omega^{-1}x)$  avec  $a_0, a_k, b_k$  rationnels]; (b)  $\{x_i\}$  est "équirépartie totalement" si elle est équirépartie pour tout module entier (et par suite pour tout module rationnel); (c)  $\{x_i\}$  est "unifiante totalement" si elle est unifiante pour tout module entier (et par suite pour tout module rationnel).

Appelant "normale" toute fonction  $x(t)$  définie sur  $(0, 1)$  et pourvue d'une dérivée  $x'(t) > 0$ , continue et non-décroissante, et "suite normale" toute suite  $\{x_i(t)\}$  de fonctions normales  $x_i(t)$  telles que  $\lim_{t \rightarrow 0} x_i'(0) = +\infty$ , l'auteur démontre que: (1) si une suite de fonctions  $\{x_i(t)\}$  est normale, la suite numérique  $\{x_i(t)\}$  (où  $t$  à une valeur déterminée quelconque) est: (a) unifiante pour tout module  $\omega$  pour presque toutes les valeurs de  $t$ ; (b) unifiante totalement pour presque toutes les valeurs de  $t$ ; (2) en prenant  $x_i(t) = a_i t$  où  $a_i$  est indépendant de  $t$ , si la suite  $\{x_i(t)\}$  (qui n'est pas forcément normale) est équirépartie mod (1) presque-partout, elle est équirépartie totalement presque-partout; si elle n'est pas équirépartie mod (1) presque-partout, elle n'est équirépartie totalement que sur un ensemble de mesure nulle.

Ces résultats sont obtenus essentiellement à l'aide des lemmes suivants. (A) Étant donnée une suite infinie  $\{F_n(t)\}$  de fonctions réelles sommables sur  $(0, 1)$ , si, quel que soit  $\alpha$ , on a:  $\lim_{n \rightarrow \infty} \int_0^1 F_n(t) dt = 0$  on a presque-partout  $\liminf F_n(t) \leq 0 \leq \limsup F_n(t)$ , pourvu que les  $F_n(t)$  admettent une majorante fixe sommable [Note du reviewer: il suffit d'une condition sensiblement moins stricte]. (B) Si la suite de fonctions  $\{x_i(t)\}$  est normale, on a:  $\lim_{t \rightarrow 0} \int_0^1 P[x_i(t)] dt = [P]$  pour toute fonction  $P(x)$  périodique de période 1 et sommable. L'auteur termine par des compléments et des exemples. *R. Fortet* (Caen).

**Ammann, A.** Sur les répartitions des suites de nombres réels. Résumé d'une thèse présentée à l'Université de Genève. Comment. Math. Helv. 21, 327-331 (1948). Cf. the preceding review.

**Skolem, Th.** A proof of the algebraic independence of  $e$  and  $e^{\sqrt{-1}d}$ ,  $d$  positive integer, with another proof of the irrationality of  $\log x$  and  $\operatorname{arctg} x$  for rational  $x$ . Norsk Mat. Tidsskr. 28, 97-104 (1946).

The author has given a new and simple proof for the theorem of Lindemann that the numbers  $e^{\alpha_1}, \dots, e^{\alpha_n}$  are algebraically independent if  $\alpha_1, \dots, \alpha_n$  are linearly independent algebraic numbers [Norske Vid. Selsk. Forh., Trondhjem 19, no. 12, 40-43 (1947); these Rev. 9, 413]. In the present paper he treats the special case  $n=2$ ,  $\alpha_1$  rational and  $\alpha_2$  belonging to an imaginary quadratic number field, and then his proof is almost identical with the usual proof for the transcendence of  $e$  [see, e.g., E. Landau, Vorlesungen über Zahlentheorie, v. 3, Hirzel, Leipzig, 1932, p. 92].

The second section of the paper contains proofs for the irrationality of  $\log x$  and  $\operatorname{arctg} x$  for rational values of  $x$  (if naturally the trivial cases  $x=0, 1$  or  $x=0$ , respectively, are excluded). The starting point is the formula of Hermite,

$$\int_{-1}^1 (1-x^2)^n \cos ax dx = n! x^{-2n-1} (P \sin x + Q \cos x),$$

where  $P$  and  $Q$  are polynomials in  $x$  with integral coefficients. *J. Popken* (Utrecht).

**Salzer, H. E.** Further remarks on the approximation of numbers as sums of reciprocals. Amer. Math. Monthly 55, 350-356 (1948).

The author begins with a bibliography of the  $R$ -expansions, which indeed go back to Lambert [1770]. Then he continues his former publication [same Monthly 54, 135-142 (1947); these Rev. 8, 534] and obtains the theorem: if  $p/q$  is an approximation of  $x$  obtained by the  $R$ - or  $\bar{R}$ -expansion, respectively, then the remainder  $x - p/q$  is less than  $1/q$  or absolutely less than  $1/2q$ , respectively.

*E. Bodewig* (The Hague).

**Hinčin, A. Ya.** Regular systems of linear equations and a general problem of Čebyšev. Izvestiya Akad. Nauk SSSR. Ser. Mat. 12, 249-258 (1948). (Russian)

Let  $L_j = \sum_{i=1}^m \theta_{ij} x_i - y_j$ ,  $(1 \leq j \leq n)$ ,  $x = \max |x_i|$ , where  $\theta_{ij}$  are  $mn$  real constants and  $x_i$  and  $y_j$  are  $m+n$  integer variables. The set of forms  $L_j$  is called a "Čebyšev set" if to every set of real numbers  $\alpha_j$  there corresponds a  $\Gamma$  such that the set of inequalities

$$(I) \quad |L_j - \alpha_j| < \Gamma x^{-m/n} \quad (1 \leq j \leq n)$$

is soluble in  $x_i$  and  $y_j$  for arbitrarily large values of  $x$ . The set is called "regular" if for some positive  $\delta$  and some arbitrarily large values of  $t$  the inequalities

$$(II) \quad |L_j| \leq t^{-1} \quad (1 \leq j \leq n), \quad 0 < x \leq t^{m/n},$$

are not soluble. Result: a set is a Čebyšev set if and only if it is regular. The proof of "if" is simple, using Mordell's device of replacing  $\alpha_j$  by  $u\alpha_j$  in (I), where  $u$  is a new integer variable; the modified (I) will have solutions in  $x_i, y_j, u$  by Minkowski's classical theorem on convex regions, with  $u$  not exceeding a bound depending on  $\Gamma$  only; the case  $u=0$  is excluded by the condition of regularity, and the case  $u \neq 0$  leads back to the unmodified (I) with a new value of  $\Gamma$ . The proof of "only if" is longer, but equally easy apart from one interesting twist; it is proved first that a set is a Čebyšev set only if the transposed set of forms (with an obvious definition) is regular; then the complete proof of equivalence follows the cycle (Čebyšev → transposed regular → transposed Čebyšev → regular → Čebyšev). These results had been proved previously by the author in the case  $n=1$  only [Acta Arith. 2, 161-172 (1937)]. *F. J. Dyson*.

**Chalk, J. H. H.** On the positive values of linear forms. II. Quart. J. Math., Oxford Ser. 19, 67-80 (1948).

[For part I cf. the same J., Oxford Ser. 18, 215-227 (1947); these Rev. 9, 413.] Let  $L_1, \dots, L_n$  be  $n$  homogeneous forms in  $n \geq 2$  variables  $u_1, \dots, u_n$ , whose determinant  $\Delta$  is not zero. The author investigates the question whether there are infinitely many sets of integers  $u_1, \dots, u_n$ , such that  $L_1 > 0, L_2 > 0, \dots, L_n > 0$  and  $L_1 L_2 \dots L_n \leq |\Delta|$ . He proves that the answer is positive for all sets of forms for which not all of the  $n$  sets  $S_i$  of  $n-1$  equations  $L_j = 0$  ( $j=1, \dots, n$ ;  $i \neq j$ ;  $i=1, 2, \dots, n$ ) are soluble in integers not all zero.

The proof depends on a similar theorem for inhomogeneous forms which is deduced by the author from Kronecker's theorem. The author points out the relationship between his results and some theorems of K. Mahler [Proc. London Math. Soc. (2) 49, 128-157 (1946); Nederl. Akad. Wetensch., Proc. 49, 331-343, 444-454, 524-532, 622-631 = Indagationes Math. 8, 200-212, 299-309, 343-351, 381-390 (1946); these Rev. 8, 12].

J. F. Koksma (Amsterdam).

Dyson, F. J. On the product of four non-homogeneous linear forms. *Ann. of Math.* (2) 49, 82-109 (1948).

A famous conjecture, attributed to Minkowski, asserts: Let  $y_i = a_{i1}x_1 + \dots + a_{in}x_n$  ( $i = 1, \dots, n$ ) be  $n$  forms with real coefficients and the determinant  $\Delta \neq 0$ ; let  $\eta_1, \dots, \eta_n$  be real numbers. Then there are  $n$  integers  $x_1, \dots, x_n$  such that  $|y_1 - \eta_1| \dots |y_n - \eta_n| \leq 2^{-n}|\Delta|$ ; the sign of equality is necessary if and only if  $y_i = p_i v_i$ ,  $\eta_i = p_i(m_i + \frac{1}{2})$ , where  $v_1, \dots, v_n$  are unimodular forms in  $x_1, \dots, x_n$  with integer coefficients,  $p_1, \dots, p_n$  are real numbers whose product is  $\Delta$  and  $m_1, \dots, m_n$  are integers. This conjecture has been proved for  $n=2$  by Minkowski and for  $n=3$  by Remak [Math. Z. 17, 1-34 (1923); 18, 173-200 (1923)] and later much more simply by Davenport [J. London Math. Soc. 14, 47-51 (1939)]. An attempt to prove it for  $n=4$  has been made by Hofreiter [Monatsh. Math. Physik 40, 351-392, 393-406 (1933)]. Here a complete proof is given for  $n=4$ .

The result is an easy consequence of the following two theorems. (A) Let  $S$  be a four-dimensional space with Cartesian coordinates  $y_1, \dots, y_4$  and origin  $O$ . Let  $L$  be a set of points in  $S$  with the following properties: (i)  $O$  non- $L$ ; (ii) every bounded set contains only a finite number of points of  $L$ ; (iii) given any one  $A$  of the coordinate axes and any  $\delta > 0$ ,  $L$  contains a point the distance of which from  $A$  is less than  $\delta$ . Then there are numbers  $\lambda_i > 0$  ( $i = 1, \dots, 4$ ) such that the ellipsoid  $\lambda_1 y_1^2 + \dots + \lambda_4 y_4^2 = 1$  has no points of  $L$  inside it but has four points of  $L$  on its surface not lying in a three-dimensional subspace through  $O$ . (B) Let  $L$  be a lattice with determinant  $\Delta \neq 0$  in four-dimensional space which contains five points  $O, P_1, P_2, P_3, P_4$  not lying in a three-dimensional subspace and such that  $OP_1 = OP_2 = OP_3 = OP_4$  and  $OP \geq OP_1$  for every point  $P \in L$  other than  $O$ . Then every hypersphere of radius  $|\Delta|^{\frac{1}{4}}$  contains a point of  $L$  in its interior or on its surface; the supplement "or on its surface" is necessary if and only if  $L$  is a rectangular cubic lattice and the hypersphere has its centre at the centre of one of the cells of  $L$ . Theorem (A) is a purely geometrical theorem; its proof is based on a topological lemma which has been proved by the author in a separate paper [same Ann. 49, 75-81 (1948); these Rev. 10, 55]. The proof of (B) belongs to the geometry of numbers and uses the classical theorem on the minimum of a positive definite quaternary form. The author observes that he has not been able to find any reference to the conjecture in Minkowski's papers. [On p. 108, line 9 from below, read  $\theta_{k,1}$  instead of  $\varphi_{k,1}$ .]

V. Jarník (Prague).

Macbeath, A. M. The minimum of an indefinite binary quadratic form. *J. London Math. Soc.* 22 (1947), 261-262 (1948).

A simple proof is given of the following theorem due to Korkine and Zolotareff [Math. Ann. 6, 366-389 (1873)]. Let  $f(x, y) = ax^2 + bxy + cy^2$ , where  $a, b, c$  are real and  $d = b^2 - 4ac > 0$ . Then integers  $x, y$  exist such that

$|f(x, y)| \leq \sqrt{(\frac{1}{4}d)}$ . The equality sign is necessary if and only if  $f(x, y)$  is equivalent to a multiple of  $x^2 + xy - y^2$ .

H. S. A. Potter (Aberdeen).

Varnavides, P. Non-homogeneous binary quadratic forms.

I, II. *Nederl. Akad. Wetensch., Proc.* 51, 396-404, 470-481 = *Indagationes Math.* 10, 142-150, 164-175 (1948).

Minkowski zeigte: Sind  $\xi = ax + by$ ,  $\eta = cx + dy$  zwei Linearformen mit der Determinante  $\Delta = ad - bc \neq 0$ ,  $a$  und  $b$  beliebige reelle Zahlen, so gibt es ganze Zahlen  $x$  und  $y$ , dass  $|(\xi - a)(\eta - b)| \leq \frac{1}{4}|\Delta|$  ist, und dabei lässt sich die Konstante auf der rechten Seite nicht verbessern. In der vorliegenden Arbeit werden nur spezielle Linearformen untersucht, indem  $\xi$  eine beliebige ganze Zahl aus dem quadratischen Körper  $k(\sqrt{2})$  und  $\xi'$  ihre Konjugierte sein soll. Dann wird gezeigt: Für beliebige reelle Zahlen  $a, b$  gibt es eine solche Zahl  $\xi$  mit  $|(\xi - a)(\xi' - b)| \leq \frac{1}{4}$ . Dabei gilt Gleichheit nur für  $a = \pm 2^{-1} + \xi_0$ ,  $b = \mp 2^{-1} + \xi'_0$  und beliebiges ganzzahliges  $\xi_0$  aus  $k(\sqrt{2})$ ; und weiter: Sind  $a, b$  von dieser Form, so besteht die Ungleichung  $|(\xi - a)(\xi' - b)| \leq \frac{1}{4}$  für alle ganzen  $\xi$  aus  $k(\sqrt{2})$  und hier tritt die Gleichheit für unendlich viele Werte von  $\xi$  ein.

Sind hingegen  $a, b$  von der Form  $a = (\omega/\alpha_n) + \xi_0$ ,  $b = (\omega'/\alpha_n') + \xi'_0$  oder  $a = (\omega/\alpha_n) + \xi_0$ ,  $b = (\omega'/\alpha_n) + \xi'_0$ , wobei  $\omega$  eine Einheit von  $k(\sqrt{2})$ ,  $\xi_0$  eine ganze Zahl aus  $k(\sqrt{2})$ ,  $n$  eine ungerade positive ganzzahlige Zahl und  $\alpha_n = 2^{1/(\tau^{n+1}-1)}/(\tau^n+1)$ ,  $\tau = 1 + \sqrt{2}$  bedeute, dann gibt es eine ganze Zahl  $\xi$  mit  $|(\xi - a)(\xi' - b)| \leq 1/(\alpha_n \alpha_n')$ , und wenn  $a, b$  nicht von einer der angegebenen Spezialformen ist, ergibt sich: Untere Grenze  $(|(\xi - a)(\xi' - b)|) \leq 1/(2\tau) = 1/4.828 \dots$ . Sind jedoch  $a, b$  von der zuletzt angegebenen speziellen Gestalt, dann gilt  $|(\xi - a)(\xi' - b)| \leq 1/(\alpha_n \alpha_n')$  für alle ganzzahligen  $\xi$  aus  $k(\sqrt{2})$  und die Gleichheit tritt hier für unendlich viele ganzzahlige  $\xi$  ein.

Diese Ergebnisse und die Beweise derselben sind analog früheren Ergebnissen von H. Davenport [Nederl. Akad. Wetensch., Proc. 50, 378-389, 741-749, 909-917 = *Indagationes Math.* 9, 236-247, 351-359, 420-428 (1947); diese Rev. 8, 565; 9, 412] die sich auf die ganzen Zahlen  $\xi$  aus  $k(\sqrt{5})$  beziehen.

T. Schneider (Göttingen).

Cassels, J. W. S. On a theorem of Rado in the geometry of numbers. *J. London Math. Soc.* 22 (1947), 196-200 (1948).

Sei  $f(x)$  eine beschränkte nicht-negative Vektorfunktion der  $n$  reellen Veränderlichen  $x = [x_1, \dots, x_n]$  im  $R_n$  und es verschwindet  $f(x)$  außerhalb eines beschränkten Bereiches des  $R_n$ . Es sei  $\lambda$  eine nicht-singuläre Matrix;  $\xi$  durchläuft alle Gitterpunkte eines gegebenen Gitters der Determinante  $D > 0$ . Ist nun  $f(\lambda x - \lambda y) \geq \min \{f(x), f(y)\}$ , so hat R. Rado [J. London Math. Soc. 21, 34-47 (1946); diese Rev. 8, 444] gezeigt, dass die Ungleichung

$$f(0) + \frac{1}{2} \sum_{\xi \neq 0} f(\xi) \geq (\|\lambda\|/D) \int_{R_n} f(x) dx = (\|\lambda\|/D)v$$

besteht. Hier wird bewiesen: Gelten für  $f(x)$  die Bedingungen  $f(\lambda x - \lambda y) \geq k \{f(x) + f(y)\}$  für positives  $k \leq \frac{1}{2}$  und alle Vektoren  $x$  und  $y$  im  $R_n$  und ferner  $f(0) = \max f(x)$ , so ist

$$f(0) + \frac{1}{2k} \sum_{\xi \neq 0} f(2\xi) \geq \frac{\|\lambda\|v}{D} + \frac{1}{\|\lambda\|Dv} \sum_{\xi \neq 0} \left| \int f(\lambda^{-1}x) e^{2\pi i \langle x, \xi \rangle} dx \right|^2$$

erfüllt. Die Beweise benutzen eine Methode von C. L. Siegel [Acta Math. 65, 307-323 (1935)], die auf der Besselschen Ungleichung bei  $n$ -fachen Fourier-Reihen beruht.

T. Schneider (Göttingen).

## ANALYSIS

Niven, Ivan. Note on a paper by L. S. Johnston. Amer. Math. Monthly 55, 358 (1948).

In this note on a paper by L. S. Johnston [same vol., 65-70 (1948); these Rev. 9, 416] the author gives a constructive one-to-one correspondence between the positive integers and the rational numbers. The one-to-one correspondence between the positive integers and all positive rational numbers is effected by letting 1 correspond to 1 and  $n+1$  to  $f(n)$ , where  $f(n) = \prod_{j=1}^{r+1} p_j^{a_j}$ ,  $p_j = -a_j/2$  for even  $a_j$ ,  $p_j = (a_j+1)/2$  otherwise,  $a_j$  ( $j=2, 3, \dots, r$ ) denotes the number of 1's between the  $(j-1)$ th and the  $j$ th zeros, counted from the right, in the representation of the positive integer  $n$  in the binary scale of notation,  $r$  denotes the number of zeros occurring in this representation,  $a_1$  and  $a_{r+1}$  are the number of consecutive 1's on the right and left ends of the representation, and  $p_1, p_2, \dots$  are the consecutive primes. The one-to-one correspondence between the positive integers and all rational numbers is obtained from the correspondences 1 to 0, 2 to 1, 3 to  $-1$ ,  $2n+2$  to  $f(n)$ ,  $2n+3$  to  $-f(n)$ , where  $n$  ranges over all positive integers.

W. H. Gage (Vancouver, B. C.).

Čebotarëv, N. G. On the expression of Abelian integrals by means of elementary functions. Uspehi Matem. Nauk (N.S.) 2, no. 2(18), 3-20 (1947). (Russian)

This is an expository paper. It starts with a proof that if  $y(x)$  is an algebraic function of  $x$  whose integral is an algebraic combination of  $x$  and of logarithms of algebraic functions, the integral is an algebraic function plus a sum of constant multiples of logarithms of algebraic functions. This is part of Liouville's famous theorem on algebraic functions with elementary integrals. There follows a proof of Abel's theorem to the effect that, when  $y(x)$  can be integrated as above, the integral can be expressed in such a way that all algebraic functions found in it are rational in  $x$  and  $y(x)$ . A derivation is given of Čebyšev's theorem on the cases of integrability of  $x^m(1+x^n)^p$ . The treatment is similar to that in the reviewer's monograph [Integration in Finite Terms, Columbia University Press, New York, 1948; these Rev. 9, 573]. Abel's theorem on elementary hyperelliptic integrals, in which the periodicity of a continued fraction is the integrability criterion, is next derived. A partial discussion is given of Zolotareff's test for the elementary integrability of certain elliptic integrals.

J. F. Ritt (New York, N. Y.).

Szarski, J. Sur une méthode d'approximation des fonctions. Ann. Soc. Polon. Math. 20 (1947), 121-125 (1948).

L'auteur approche une fonction continue  $f(x)$  par le procédé de régularisation par moyennes:  $f_n(x)$  est la moyenne de  $f$  dans un cube de côté  $1/n$  de centre  $x$ ; pour  $n \rightarrow \infty$ ,  $f_n$  converge vers  $f$ . Tous les résultats de cet article sont connus.

L. Schwartz (Nancy).

Christov, Christo. Sur un problème de M. Pompeiu. Mathematica, Timișoara 23, 103-107 (1948).

Let  $C$  be a circle of fixed radius. Then the equation (1)  $\iint_C F(x, y) dx dy = 0$  is satisfied for all positions of  $C$  in the  $(x, y)$ -plane by functions of the form (2)  $F(x, y) = \sin(ax+by)$ , where  $a$  and  $b$  are suitable constants; there is an infinitude of linearly independent solutions of the form (2). On the other hand, D. Pompeiu has shown [Bull. Sci. Math. (2)

53, 328-332 (1929)] that if  $C$  is a square with side of fixed length, and the continuous function  $F(x, y)$  has a unique limit as  $x^2+y^2 \rightarrow \infty$ , then (1) holds for all positions of the square  $C$  in the  $(x, y)$ -plane if and only if  $F(x, y) = 0$ . The author now establishes the result of Pompeiu without the supplementary restriction that the continuous function  $F(x, y)$  have a limit as  $x^2+y^2 \rightarrow \infty$ .

E. F. Beckenbach.

Lelong, Pierre. Sur l'approximation des fonctions de plusieurs variables au moyen des fonctions polyharmoniques d'ordres croissants. C. R. Acad. Sci. Paris 227, 26-28 (1948).

The author outlines (without proofs) a theory of quasi-analytic functions of several variables in analogy with S. Bernstein's for one variable [Leçons sur les Propriétés Extrêmales et la Meilleure Approximation des Fonctions Analytiques d'une Variable Réelle, Paris, 1926]. Let  $z = x + iy = (z_1, \dots, z_p)$  be a complex vector, let  $D$  be a domain in the  $x$ -space and  $H(D)$  the largest domain in the  $z$ -space, containing  $D$ , inside which  $\sum (z_j - \xi_j)^2 \neq 0$ , when  $\xi$  is an arbitrary boundary point of  $D$ . Let  $K$  be a compact subset of  $H(D)$ . Then there exists a positive number  $\alpha = \alpha_p(K)$  and further there corresponds to  $A > 0$  a positive number  $\tau$ , such that a function  $U_n(x)$ , which is  $n$ -harmonic in  $D$  and satisfies  $|U_n(x)| \leq A$  in  $D$ , will satisfy  $|U_n(x)| \leq A\alpha(1+\tau)^{2n-2}(n+p/2)^p \log n$  in  $K$ . If a function  $f(x)$  in  $D$  satisfies  $|f(x)| \leq m_0$ ,  $|\Delta^{(n)} f(x)| \leq m_n$ , the author finds upper bounds for all partial derivatives of order not exceeding  $2n-1$ . Let  $n_k$  be a sequence of positive integers and let  $\rho$  be a number between 0 and 1. A function  $f(x)$  is said to belong to the class  $\{n_k, \rho\}$  in  $D$  if every point of  $D$  is contained in a neighborhood in which  $f(x)$ , for all values of  $k$ , can be approximated uniformly with the accuracy  $A\rho^{nk}$  by an  $n_k$ -harmonic function. The class  $\{n_k, \rho\}$  is quasi-analytic in the sense that two functions of the class will be identical if they are identical in an arbitrary open subset of  $D$ .

H. Tornehave (St. Johns, Que.).

Verblunsky, S. Additional note on two moment problems for bounded functions. Proc. Cambridge Philos. Soc. 44, 140-142 (1948).

In a previous paper, referred to as (I) [same Proc. 42, 189-196 (1946); these Rev. 8, 153], the author established a correspondence, valid both ways, between the class of functions  $f(x) \in L(-\infty, \infty)$ ,  $0 \leq f(x) \leq 1$ , and the class of bounded nondecreasing functions  $\sigma(x)$ , connected by relation

$$\exp \left( \int_{-\infty}^{\infty} \frac{f(x)}{\xi - x} dx \right) = 1 + \int_{-\infty}^{\infty} \frac{d\sigma}{\xi - x}, \quad \Re \xi > 0.$$

This result, and some of its refinements, were applied in (I) to show that if numbers  $s_0, s_1, \dots, s_{n-1}$  and  $t_0, t_1, \dots, t_{n-1}$  are connected by the relation

$$\exp \left( \sum_0^{n-1} s_k / z^{k+1} \right) = 1 + \sum_0^{n-1} t_k / z^{k+1} + \dots,$$

then there will exist a function  $f(x)$ , where  $0 \leq f \leq 1$ , such that  $s_k = \int_{-\infty}^{\infty} x^k f(x) dx$ ,  $k=0, \dots, n-1$ , if and only if  $t_k = \int_{-\infty}^{\infty} x^k d\sigma(x)$ ,  $k=0, \dots, n-1$ , where  $d\sigma(x) \geq 0$ , provided  $n$  is odd. Using further properties of the above correspondence  $f \leftrightarrow \sigma$ , this result is now established also for even  $n$ .

I. J. Schoenberg (Philadelphia, Pa.).

Fuchs, W. H. J. On a generalization of the Stieltjes moment problem. Bull. Amer. Math. Soc. 52, 1057-1059 (1946).

The generalized Stieltjes moment problem is as follows. Given the infinite sequence  $\lambda_0=0 < \lambda_1 < \dots < \lambda_n \rightarrow \infty$  of exponents and the sequence of moments  $\mu_n$ , to find a nondecreasing function  $\alpha(t)$  satisfying the equations (1)  $\int_0^\infty t^n d\alpha(t) = \mu_n$  ( $n=0, 1, \dots$ ). The problem (1) is said to be determined if, disregarding additive constants, it has at most one solution. The author succeeds in proving the following theorem. Assume that there is a  $c > 0$  such that  $\lambda_{n+1} - \lambda_n > c$  ( $n=1, 2, \dots$ ) and let  $\psi(r) = \exp\{2\sum_{n \geq 1} \lambda_n r^{-1}\}$ . If there are a nonincreasing function  $\varphi(r)$  and positive constants  $A$  and  $a$  such that  $\psi(r) > A(r/\varphi(r))^a$  and if

$$\sum_{n=1}^{\infty} \frac{\lambda_n - \lambda_{n-1}}{\mu_n^{1/(a\lambda_n)} \varphi(\lambda_{n-1})} = \infty,$$

then the problem (1) is determined. In the classical case when  $\lambda_n = n$  we may choose  $a=2$ ,  $\varphi(r)=1$  and the theorem yields Carleman's criterion  $\sum \mu_n^{-1/(2n)} = \infty$ . The theorem does not imply nor is it implied by results of Boas [Trans. Amer. Math. Soc. 46, 142-150 (1939); these Rev. 1, 13]. The coefficient 2 in the definition of  $\psi(r)$  was left out in the paper by a misprint. *I. J. Schoenberg* (Philadelphia, Pa.).

Mandelbrojt, Szolem. Quelques considérations sur le problème des moments. C. R. Acad. Sci. Paris 226, 862-864 (1948).

Let  $\{i_n\}$  be a sequence of increasing integers,  $i_0=0$ . Put, for any sequence  $\{\alpha_n\}$ ,  $\alpha_n^* = \sup_{p \geq n} \alpha_p$ . Write

$$\tau_n = (\log \mu_{n+1} - \log \mu_n) / (i_{n+1} - i_n)$$

( $n > 0$ ),  $\tau_0 = 0$ ,  $k_n = \frac{1}{2} \sum_{n=1}^m (\tau_n - \tau_{n-1}) (i_n/n)$ . ( $k_n = \infty$ , if an  $(i_n/n)^* = \infty$ ). (1) The moment problem (S)  $\int_0^\infty t^n dV(t) = \mu_n$  has at most one distribution function  $V(t)$  as solution, if  $\sum_{n=1}^{\infty} (i_{n+1} - i_n) e^{-k_n} = \infty$ . A corresponding result for the moment problem (H)  $\int_0^\infty t^n dV(t) = \mu_n$  is (2). Let  $\{j_n\}$  contain an infinite subsequence of even integers  $\{i_n\}$  and let  $m_n$  be the number of positive  $j_m < i_n$ . Suppose  $i_n < 2m_n$ . Write  $\mu_n = \mu_{i_n}$ ,  $k_n = \sum_{n=1}^m (\tau_n - \tau_{n-1}) (i_n / (2m_n - i_n))$ . Then the solution of (H) is unique if  $\sum_{n=1}^{\infty} (i_{n+1} - i_n) e^{-k_n} = \infty$ . For  $i_n = n$  and  $j_n = n$  the theorems include Carleman's conditions for uniqueness:  $\sum (\mu_n / \mu_{n+1})^{1/2} = \infty$  in the case of (S),  $\sum \mu_n / \mu_{n+2} = \infty$  in the case of (H). The proof of the theorems depends on the author's "fundamental inequality" [Ann. Sci. École Norm. Sup. (3) 63, 351-378 (1946), theorem F, form A; these Rev. 9, 229]. Theorem 1 neither contains nor is contained in previous results of Boas and Fuchs [cf. the preceding review]. *W. H. J. Fuchs* (Ithaca, N. Y.).

Boas, R. P., Jr., and Chandrasekharan, K. Derivatives of infinite order. Bull. Amer. Math. Soc. 54, 523-526 (1948).

The authors answer yes to the two following questions asked by Ganapathy Iyer [J. Indian Math. Soc. (N.S.) 8, 94-108 (1944); these Rev. 7, 117]. (I) If  $f^{(n)}(x) \rightarrow g(x)$  for each  $x$  in  $(a, b)$  where  $g(x)$  is finite, does  $g(x) = ke^x$ ? (II) If  $f(x)$  belongs to a quasianalytic class in  $(a, b)$  and  $\lim_{n \rightarrow \infty} f^{(n)}(x_0)$  exists for a single  $x_0$ , does  $\lim_{n \rightarrow \infty} f^{(n)}(x)$  exist for every  $x$  in  $(a, b)$ ? The answer to (I) follows from the fact that with the hypotheses of (I)  $g(x)$  is analytic. From the hypotheses of (II) it also follows that  $f(x)$  is analytic. With conditions on  $\lambda_n$  the authors can replace  $f^{(n)}(x) \rightarrow g(x)$  by  $f^{(n)}(x) / \lambda_n \rightarrow g(x)$  and still have interesting results. [Re-

viewer's remark. There is an error in the proof of (II). The authors start from the property (a)  $\{M_n^*\}$  being the exponentially regularized sequence, by logarithms, of  $\{M_n\}$ , the sequence  $M_{n+1}^* / M_n^*$  is nondecreasing. It would follow from it that, for each  $\{M_n\}$  (except for the trivial case  $M_n = 0$ ), (b)  $C(1) \subset C\{M_n\}$ . Neither of these assertions (a), (b) is absolutely correct, the second being true, however, for the whole axis. But the authors' theorems are all correct, since the assertion (b) can be replaced by another which gives the results.] *S. Mandelbrojt* (Houston, Tex.).

Popoviciu, Tiberiu. Notes sur les généralisations des fonctions convexes d'ordre supérieur. III. Acad. Roum. Bull. Sect. Sci. 24, 409-416 (1943).

For the author's generalization of functions of order  $n$  to functions of order  $(n|k)$ , one should consult his previous article [Disquisit. Math. Phys. 1, 35-42 (1940); these Rev. 9, 14]. It is now shown that any given function of order  $n$  by segments is of a definite order  $(n|k)$ , where  $k$  is the number of variations of sign in a certain sequence, and that every function of order  $(n|k)$ , for any  $k$ , is of order  $n$  by segments. *E. F. Beckenbach* (Los Angeles, Calif.).

Thorin, G. O. Convexity theorems generalizing those of M. Riesz and Hadamard with some applications. Comm. Sem. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.] 9, 1-58 (1948).

Cette thèse de l'Université de Lund expose d'abord nombre de résultats connus relatifs à la convexité du maximum d'une fonction sousharmonique ou plurisousharmonique (l'auteur dit: bisousharmonique) sur un ensemble  $E$ , fonction croissante d'un paramètre  $t$  convenablement choisi. A signaler [p. 17] une présentation des fonctions et des fonctionnelles sousharmoniques sur un espace vectoriel rapporté à une base. Par la suite sont envisagées surtout des applications du principe suivant, assez restreint: la fonction réelle  $f(x, y)$  étant convexe de  $x$  pour  $x \in C$  ( $C$  convexe),  $L$  désignant un opérateur linéaire d'une famille  $K$ , et  $u$  appartenant à la région convexe  $\Gamma = L^{-1}(C)$ , alors  $M(u, K) = \limsup f(Lu, y)$  est une fonction convexe de  $u$  dans  $\Gamma$ . Des propriétés bien connues des modules des fonctions analytiques de  $p$  variables sont retrouvées ainsi.

La dernière partie du mémoire donne des démonstrations d'inégalités connues en les rattachant à des propriétés simples de convexité des fonctions analytiques ou des fonctions plurisousharmoniques. C'est le cas pour le théorème de M. Riesz exprimant que le maximum  $M(\alpha, \beta)$  de la forme  $\sum_{i, j=1}^n a_{ij} x_i y_j$ , pour  $\sum_i p_i |x_i|^{1/p} \leq 1$ ,  $\sum_i \sigma_i |y_i|^{1/p} \leq 1$  ( $p_i > 0$ ,  $\sigma_i > 0$ ), est une fonction convexe de  $\alpha, \beta$  pour  $\alpha \geq 0$ ,  $\beta \geq 0$ . L'auteur l'étend aux formes multilinéaires, puis l'étudie dans le champ réel où il ne vaut que pour  $\alpha + \beta \geq 1$ ,  $\alpha \geq 0$ ,  $\beta \geq 0$ . La même méthode est appliquée avec succès à la démonstration de l'inégalité de Hölder, à celle d'un théorème de Hardy et Littlewood majorant la somme  $\sum_n (|n|+1)^{p-2} |a_n|^p$  relative aux coefficients de Fourier d'une fonction de classe  $L^p$  ( $1 < p \leq 2$ ) et, avec plus de difficultés, à des majorations d'intégrales multiples dues aux mêmes auteurs. *P. Lelong* (Lille).

Laguardia, Rafael. On the extension of an inequality of Tchebycheff. Facultad de Ingeniería Montevideo. Publ. Inst. Mat. Estadística 1, 159-161 (1947). (Spanish)

The paper also appeared in Bol. Fac. Ingen. Montevideo 3 (Año 10), 229-231 (1946); these Rev. 8, 504.

*Calculus*

\*Tricomi, Francesco. *Lezioni di Analisi Matematica*. Parte Prima. 6th ed. CEDAM, Padova, 1948. xii+337 pp. 1800 Lire.

This is a slight revision of the 4th edition [1939]; the 5th edition [1943] appeared only in lithographed form.

Boggio, Tommaso. *Sur une proposition de M. Pompeiu*. *Mathematica*, Timișoara 23, 101-102 (1948).

If  $f(x)$  and  $\varphi(x)$  are differentiable for  $x_1 \leq x \leq x_2$  and  $\varphi'(x) \neq 0$  for any  $x$  in that interval, then

$$\frac{\varphi(x_1)f(x_2) - \varphi(x_2)f(x_1)}{\varphi(x_1) - \varphi(x_2)} = f(x_0) - \frac{\varphi(x_0)}{\varphi'(x_0)}f'(x_0)$$

with  $x_1 < x_0 < x_2$ . The case  $\varphi(x) = x$  was given by Pompeiu [same journal 22, 143-146 (1946); these Rev. 8, 15].

P. Civin (Eugene, Ore.).

Cattaneo, Paolo. *Esercizi sugli iperspazi*. *Matematiche*, Catania 1, 104-112 (1946).

The author computes some volumes bounded by portions of hyperquadrics and hyperplanes. R. P. Boas, Jr.

Willis, H. F. *A formula for expanding an integral as a series*. *Philos. Mag.* (7) 39, 455-459 (1948).

The integral  $\int_{a_0}^{b_0} f(x)F(x)dx$  has the formal expansion  $\sum_{n=0}^{\infty} (-1)^n A_n f^{(n)}(c)$  if  $\int_{a_0}^{b_0} F(x)e^{-\alpha(x-c)}dx$  has the expansion  $\sum_{n=0}^{\infty} A_n \alpha^n$ ; this is obtained by expanding  $f(x)$  in a Taylor series around  $x=c$ , substituting in  $\int_{a_0}^{b_0} f(x)F(x)e^{-\alpha(x-c)}dx$  and letting  $\alpha \rightarrow 0$ . Several illustrations are given but no attempt is made to give conditions under which the formal result will have a meaning. R. P. Boas, Jr. (Providence, R. I.).

Rizzoni, Walter. *Costruzione effettiva di sferule di Pizzetti*. *Matematiche*, Catania 1, 150-162 (1946).

The author constructs some elementary solutions  $\varphi(x)$  of  $\int_0^x z^n \varphi(z)dz = 0$ . Such a function represents the density of a "sferula," which is a sphere made up of concentric layers of positive or negative density so that it has zero external attraction. R. P. Boas, Jr. (Providence, R. I.).

Kuz'min, R. O. *On Čebyšev's formula for multiple integrals*. *Doklady Akad. Nauk SSSR* (N.S.) 61, 437-439 (1948). (Russian)

This paper contains a proof of P. L. Čebyšev's first published theorem, which deals with multiple integrals [J. Math. Pures Appl. (1) 8, 235-238 (1843); Oeuvres, v. 1, St. Pétersbourg, 1899, pp. 3-6]. The author states that to the best of his knowledge no proof of this theorem has previously been published. H. P. Thielman (Ames, Iowa).

\*Monteiro de Camargo, J. O. *Cálculo Vectorial*. [Vector Calculus]. Editôra Renascença S. A., São Paulo, 1946. xiv+164 pp.

This pleasant little book is founded on lectures given by the author in the Escola Politécnica of the University of São Paulo. There are no applications other than geometrical but the author states that the book is intended to give the minimum basic knowledge necessary for the engineering sciences. The development is simple, logical and exact. Contents: I, Vectors; II, Multiplication; III, Algebraic vector equations; IV, Localised vectors; V, Coplanar vectors; VI, Vector functions of a single real variable; VII, Derivatives; VIII, Curves; IX, Vector functions of several

real variables; X, Surfaces; XI, Linear transformations; XII, Scalar fields, grad; XIII, Vector fields; XIV, Curl; XV, Div; XVI, Irrotational and solenoidal fields; XVII, Products of the operators grad, div, curl; XVIII, Integrals of vectors; XIX, Theorems of Gauss and Stokes.

L. M. Milne-Thomson (Greenwich).

Bartlett, A. C. *A form of Laplace's equation*. *Philos. Mag.* (7) 39, 326-327 (1948).

The author derives the formula  $\operatorname{div} \operatorname{grad} V = V_{nn} + KV_n$ , where  $n$  denotes the normal derivative and  $K$  is the sum of the principal curvatures of  $V$ . I. Opatowski.

*Theory of Sets, Theory of Functions of Real Variables*

Kurepa, Đuro. *Sur le continu mathématique*. *Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II*, 1, 112-125 (1946). (Croatian. French summary) Expository article.

Cuesta, N. *Dense perfectly ranked ordering*. *Revista Mat. Hisp.-Amer.* (4) 8, 57-71 (1948). (Spanish)

Let  $(E, \prec)$  be an ordered set. Two elements of  $E$  are of the same rank if there is an automorphism mapping one onto the other; the set of all elements of the same rank is termed a rank class. The ordering is said to be "ranked" if each rank class is an interval and "perfectly ranked" if each reduces to a single element. The present paper continues the author's studies of the classification of order types by showing that there exists a dense ordering which is also perfectly ranked; the author asks whether such an ordering can also be continuous, and in particular whether the completion of his example is still perfectly ranked.

R. C. Buck (Providence, R. I.).

Komm, Horace. *On the dimension of partially ordered sets*. *Amer. J. Math.* 70, 507-520 (1948).

The dimension of a partial order  $P$  is [Dushnik and Miller, same J. 63, 600-610 (1941); these Rev. 3, 73] the smallest cardinal number  $n$  such that there exist  $n$  linear orders with  $x < y$  in  $P$  if and only if  $x < y$  in each of the linear orders. The dimension of the set-inclusion ordering of the set of all subsets of a set  $S$  is the power of  $S$ . A partial order  $P_n$  on the set  $E_n$  of all sequences of  $n$  real numbers ( $n$  finite or denumerable) is defined such that the dimension of  $P_n$  is  $n$ , and any  $n$ -dimensional partial order is order-isomorphic to  $P_n$  applied to some subset of  $E_n$ . Also defined is the  $\alpha$ -dimension of a partial order, by requiring that the linear orders be on a subset of a set of order type  $\alpha$ . Among other things, it is shown that the dimension sometimes differs from the  $\alpha$ -dimension. The study of  $\alpha$ -dimension is mostly restricted to the case of  $\alpha$  equal to the order type of the continuum. P. M. Whitman (Silver Spring, Md.).

Sikorski, Roman. *On the Cartesian product of metric spaces*. *Fund. Math.* 34, 288-292 (1947).

Let  $\mathfrak{X}$  be an arbitrary metric space and let  $\mathfrak{Y}$  be a locally separable metric space. If  $D(Z)$  denotes the set of points of the space  $\mathfrak{Z}$  at which the set  $Z \subset \mathfrak{Z}$  is of second category, then, for  $X \subset \mathfrak{X}$  and  $Y \subset \mathfrak{Y}$ ,  $D(X \times Y) = D(X) \times D(Y)$ . Similar results are given for the property of Baire.

E. Hewitt (Seattle, Wash.).

**Scorza Dragoni, G.** Sull'esistenza di soluzioni per un sistema di  $n$  equazioni in  $n$  incognite. *Pont. Acad. Sci. Acta* 10, 127–134 (1946).

A demonstration of the following theorem is based on a lemma of Sperner [Abh. Math. Sem. Hamburg. Univ. 10, 1–48 (1934)]: let  $f_1, \dots, f_n$  be continuous real-valued functions of  $n$  variables defined over the unit  $n$ -cube and assume that, for each  $i$ ,  $f_i$  is nonpositive on the face of the cube which lies in  $x_i=0$  and nonnegative on the opposite face. Then there exists a point in the cube at which the functions vanish simultaneously. It is not clear who first formulated this result, although the author remarks that Miranda [Boll. Un. Mat. Ital. (2) 3, 5–7 (1940); these Rev. 3, 60] had noted its equivalence to the Brouwer fixed-point theorem.

*P. A. Smith* (New York, N. Y.).

**de Bruijn, N. G.** On the sum of a monotonic and a periodic function. *Nieuw Arch. Wiskunde* (2) 22, 241–245 (1948).

Let  $f(x)$  be a real function, defined on the whole real axis. Corresponding to an integer  $n$ ,  $A(n)$  denotes the following property of  $f(x)$ : for every set of  $n+1$  real numbers  $x_0, \dots, x_n$ , such that  $x_0 \leq \dots \leq x_n$ ,  $x_n - x_0$  an integer, and for every set of  $n$  integers  $k_1, \dots, k_n$ , the inequality  $\sum_{i=1}^n (f(x_i+k_i) - f(x_{i-1}+k_i)) \geq 0$  is satisfied. The following theorem is proved: a necessary and sufficient condition that  $f(x)$  may be expressed (on the whole real axis) as the sum of a nondecreasing function and a periodic function of period 1 is that  $f(x)$  satisfies the infinity of properties  $A(1), A(2), \dots$

*T. Viola* (Rome).

**Zahorski, Zygmunt.** Sur l'ensemble des points singuliers d'une fonction d'une variable réelle admettant les dérivées de tous les ordres. *Fund. Math.* 34, 183–245 (1947).

The principal results of this article have already been stated by the author in a part of a preliminary note [C. R. Acad. Sci. Paris 223, 449–451 (1946); these Rev. 8, 141]. A function  $f(x)$  admitting (finite) derivatives of every order at every point of the  $x$ -axis is called a function of class  $C_\infty$ . For such an  $f(x)$  at every  $x$ , one can form the Taylor series  $T(x, h) = \sum_0^\infty h^n f^{(n)}(x)/n!$ . Let  $r(x)$  be its radius of convergence. If  $r(x) = 0$ , the author calls  $x$  singular ( $P$ ) (in the sense of Pringsheim). If  $r(x) > 0$  and if there exists a number  $\delta(x)$  ( $0 < \delta(x) \leq r(x)$ ) such that  $T(x, h) = f(x+h)$  for  $|h| < \delta(x)$ , then  $x$  is a regular point. If  $r(x) > 0$ , but  $\delta(x)$  does not exist (hence there exist arbitrarily small  $h$  with  $T(x, h) \neq f(x+h)$ ), then  $x$  is called singular ( $C$ ) (in the sense of Cauchy). The set of all singular ( $P$ ) points or of all singular ( $C$ ) points is designated by  $P$  or  $C$ , respectively. The author's main result is the complete characterization of the sets  $P$  and  $C$ . It is necessary and sufficient for the sets  $P$  and  $C$  that  $P$  is a  $G_\delta$  and  $C$  is an  $F_\sigma$  of first category with  $PC = 0$ ,  $P + C = \overline{P} + \overline{C}$ . The proof of sufficiency (that is, the construction of a suitable function  $f(x)$ ) is difficult. A rather immediate consequence of the existence of a function  $f(x)$  for which every  $x$  is singular ( $P$ ) is the affirmative solution of a problem of S. M. Ulam: if  $f(x)$  is such a function, then for every analytic function  $g(x)$  the set of solutions of the equation  $f(x) = g(x)$  (in the domain of regularity of  $g(x)$ ) is at most countable, and in fact isolated. In connection with his discussions the author proves also the following theorem of A. Pringsheim [Math. Ann. 42, 153–184 (1893)] (whose proof was not correct). If for every  $x \in [a, b]$  we have  $r(x) \geq \delta > 0$ , then  $f(x)$  is regular in  $[a, b]$ . This theorem had previously been proved by V. Ganapathy Iyer [C. R. Acad.

Sci. Paris 199, 1371–1373 (1934)] and R. P. Boas [Bull. Amer. Math. Soc. 41, 233–236 (1935)]. *A. Rosenthal*.

**Banach, S.** Sur la représentation des fonctions indépendantes à l'aide des fonctions de variables distinctes. *Colloquium Math.* 1, 109–121 (1948).

Two measurable functions  $f(t), g(t)$  defined on the interval  $(0, 1)$  are "representable biaxially" if they can be written in the form  $f = \Phi[\varphi(t)], g = \Psi[\psi(t)]$ , where the transformation  $t \mapsto (\varphi(t), \psi(t))$  takes the unit interval into the unit square in such a way that the linear image of a plane Borel set has linear measure equal to the plane measure of the latter. The functions  $f$  and  $g$  will necessarily be independent if they are representable biaxially. Conversely two independent functions can be represented biaxially if their distribution functions are both continuous or both jump functions. The functions  $f = 1, g = t$  cannot be so represented.

*J. L. Doob* (Urbana, Ill.).

**Banach, S.** Sur les suites d'ensembles excluant l'existence d'une mesure. *Colloquium Math.* 1, 103–108 (1948).

To every sequence  $\{E_n\}$  of subsets of an arbitrary set  $X$  a system  $F(y)$  ( $0 \leq y \leq 1$ ) of subsets of  $X$  is assigned as follows. If  $c_n(x)$  is the characteristic function of  $E_n$  and if  $c(x) = 2 \sum_{n=1}^\infty 3^{-n} c_n(x)$  for each  $x \in X$ , then  $F(y)$  is the set of  $x$  such that  $c(x) = y$ . The following theorem is proved. Let  $E$  be the smallest  $\sigma$ -field containing  $\{E_n\}$  and  $X$ . In order that every countably additive measure on  $E$  which vanishes on all sets  $F(y)$  be identically zero, it is necessary and sufficient that the set  $Y$  of all values of  $c(x)$  be absolutely of measure 0 (i.e., that to every Borel measure  $\mu$  on  $[0, 1]$  there is a Borel set  $B \supset Y$  with  $\mu(B) = 0$ ). This theorem originated from an analysis of results of S. Banach and C. Kuratowski's paper [Fund. Math. 14, 127–131 (1929)]. Equivalent formulations are indicated by E. Marczewski who prepared this posthumous paper for publication.

*H. M. Schaefer* (St. Louis, Mo.).

**Marczewski, E., and Sikorski, R.** Measures in non-separable metric spaces. *Colloquium Math.* 1, 133–139 (1948).

The main result of this paper is a necessary and sufficient condition in order that a metric space  $X$  have the following property: for each finite Borel measure  $\mu$  in  $X$  there is a decomposition (1)  $X = N + S$ , where  $\mu(N) = 0$  and  $S$  is separable. This condition is: there is a set  $Y$  of the same power as some basis of  $X$  and such that every finite measure defined for all subsets of  $Y$  and vanishing for all one-point sets, vanishes identically. If this condition is satisfied, then the decomposition (1) exists also for every  $\sigma$ -finite Borel measure in  $X$ .

*H. M. Schaefer* (St. Louis, Mo.).

**Marczewski, E.** Indépendance d'ensembles et prolongement de mesures (résultats et problèmes). *Colloquium Math.* 1, 122–132 (1948).

"Cette note a pour but de passer en revue les théorèmes (dont les démonstrations, pour la plupart, n'ont pas été encore publiées) liant la notion d'indépendance au sens de la théorie générale des ensembles, à celle d'indépendance stochastique et apportant, en outre, des applications de ces notions au problème du prolongement des mesures et à celui de leur existence dans les produits cartésiens."

*J. L. Doob* (Urbana, Ill.).

**Špil'man, È.** On problems of the theory of measure. *Uspehi Matem. Nauk* (N.S.) 1, no. 2(12), 179–188 (1946). (Russian)

The author lists 13 unsolved problems most of which concern the existence of measures satisfying various conditions. Two typical such problems are: (1) does there exist a maximal translation invariant extension of Lebesgue measure on the line? and (2) does there exist a finite measure  $\mu$  on the class of all Borel sets of a nonseparable metric space such that  $\mu(E)=0$  whenever  $E$  is a separable Borel set?

*P. R. Halmos* (Chicago, Ill.).

**Lozinskii, S. M.** On the indicatrix of Banach. *Doklady Akad. Nauk SSSR* (N.S.) 60, 765–767 (1948). (Russian)

The author generalizes in a natural way the indicatrix of Banach  $N(x, X)$  to include all functions  $X$  in  $U_1$ , which is defined as the set of real-valued functions defined on  $[0, 1]$  such that  $\lim_{h \rightarrow 0} X(t+h)$  exists for all  $t \in [0, 1]$  and such that  $\lim_{h \rightarrow 0} X(t-h)$  exists for all  $t \in [0, 1]$ . [For Banach's definition, see S. Banach, *Fund. Math.* 7, 225–236 (1925).] The following theorems are announced without proof. (1) For any function in  $U_1$ , the integral  $\int_{-1}^1 N(x, X) dx$  is equal to the total variation of the function  $X(t)$  in the interval  $[0, 1]$ . (2) If the sequence of functions  $X_n(t)$  in  $U_1$  converges in variation to  $X_0(t)$  [see C. R. Adams and J. A. Clarkson, *Bull. Amer. Math. Soc.* 40, 413–417 (1934)], then  $\lim_{n \rightarrow \infty} \int_{-1}^1 |N(x, X_n) - N(x, X_0)| dx = 0$ . A somewhat complicated generalization of (1) is also announced.

*E. Hewitt* (Seattle, Wash.).

**Stone, M. H.** Notes on integration. I. *Proc. Nat. Acad. Sci. U. S. A.* 34, 336–342 (1948).

The author presents an outline of a theory of integration. The theory begins with a postulated "elementary integral" defined for a class of "elementary functions," and proceeds by defining, in terms of the elementary integral, an upper integral for all (extended) real functions. A pseudometric is defined in the class of all functions whose upper integral is finite; after proper identifications, the resulting function space is proved to be complete. Defining integrable function and integral, the author establishes that these concepts have many of their usual properties and proves, in particular, a theorem on term by term integration and a theorem asserting the integrability of certain (Baire) functions of integrable functions.

*P. R. Halmos* (Chicago, Ill.).

**Ayer, Miriam C., and Radó, Tibor.** A note on convergence in length. *Bull. Amer. Math. Soc.* 54, 533–539 (1948).

Let  $C = [x = x(t), y = y(t), z = z(t), a \leq t \leq b]$  be a continuous curve in space  $E_3$  and let  $L(C)$  be the length of  $C$ . If  $C_n = [x = x_n(t), y = y_n(t), z = z_n(t), a \leq t \leq b]$  is a sequence of curves, let us suppose that  $x_n \rightarrow x$ ,  $y_n \rightarrow y$ ,  $z_n \rightarrow z$  uniformly on  $(a, b)$ . Then we say that the sequence  $C_n$  converges in length to  $C$  if  $L(C_n) \rightarrow L(C)$ . We say that the sequence of continuous functions  $f_n(t)$ ,  $a \leq t \leq b$ , converges in variation to the function  $f(t)$ ,  $a \leq t \leq b$ , if  $f_n \rightarrow f$  uniformly on  $(a, b)$  and the total variation  $V(f_n)$  of  $f_n$  converges to the total variation  $V(f)$  of  $f$ . It is known that the convergence in length of  $C_n$  implies the convergence in variation of  $x_n, y_n, z_n$  to  $x, y, z$  and also the convergence in variation of any linear combination  $\alpha x_n + \beta y_n + \gamma z_n$  to  $\alpha x + \beta y + \gamma z$ . It is also known that the convergence in variation of  $x_n, y_n, z_n$  does not imply the convergence in length of  $C_n$ . The authors show that the convergence in variation of every linear combination  $\alpha x_n + \beta y_n + \gamma z_n$  implies the convergence in length of  $C_n$ . This generalizes a previous result of A. P. Morse on curves in nonparametric form.

*L. Cesari* (Princeton, N. J.).

### Theory of Functions of Complex Variables

**Carrasco, Luis Esteban.** The  $n$ th derivative of a polygenic function. *Revista Mat. Hisp.-Amer.* (4) 8, 3–11 (1948). (Spanish)

A polygenic function is a complex function  $w = f(z) = \phi(x, y) + i\psi(x, y)$ , where the real components  $\phi$  and  $\psi$  are continuous and possess continuous partial derivatives with respect to the two real independent variables  $x$  and  $y$  in a certain region of the  $z = x + iy$ -plane. The components  $\phi$  and  $\psi$  do not necessarily obey the Cauchy-Riemann equations.

The author discusses various results of Kasner and De Cicco on the first and second derivatives of a polygenic function [for a survey see *Univ. Nac. Tucumán. Revista A.* 4, 7–45 (1944); these *Rev. 7, 59*] and also gives formulas for the  $n$ th derivative  $d^n w/dz^n$  as a polygenic function of  $(x, y, y', y'', \dots, y^{(n)})$ . For  $n > 2$ ,  $d^n w/dz^n$  is linear integral in  $y^{(n-1)}$  and  $y^{(n)}$ , where the coefficient of  $y^{(n-1)}$  depends on  $y'$  and  $y''$  only and the coefficient of  $y^{(n)}$  depends on  $y'$  only. Some geometrical interpretations of these derivatives are given.

*J. De Cicco* (Chicago, Ill.).

**Bicadze, A. V.** On the so-called areolar monogenic functions. *Doklady Akad. Nauk SSSR* (N.S.) 59, 1385–1388 (1948). (Russian)

A function  $f(z) = u(x, y) + iv(x, y)$  is said to be an areolar monogenic function in its domain  $D$  of definition provided its second derivative relative to two mutually perpendicular directions is independent of the orientation of those directions. A differential characterization of areolar monogenic functions is

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - 2 \frac{\partial^2 v}{\partial x \partial y} = 0, \quad \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} = 0,$$

and an integral characterization has been given by Haskell [Bull. Amer. Math. Soc. 52, 332–337 (1946); these *Rev. 7, 381*]. Ciorănescu [Mathematica, Cluj 12, 26–30 (1936)] showed that the class of areolar monogenic functions coincides with the class of functions (1)  $f(z) = h_1(z) + zh_2(z)$ , where  $h_1(z)$  and  $h_2(z)$  are analytic in  $D$ . The author now re-establishes the characterization (1) of areolar monogenic functions, and on the basis of (1) gives alternate proofs of results of Haskell [loc. cit.] and Reade [Bull. Amer. Math. Soc. 53, 98–103 (1947); these *Rev. 8, 453*] concerning these functions.

*E. F. Beckenbach* (Los Angeles, Calif.).

**Lukomskaya, M. A.** The solution of some problems on the flow of a fluid through pores. *Akad. Nauk SSSR. Prikl. Mat. Meh.* 11, 621–628 (1947). (Russian)

The author considers the following problem: to find two analytic functions  $w_1(z) = \varphi_1 + i\psi_1$  and  $w_2(z) = \varphi_2 + i\psi_2$  having prescribed singularities and satisfying on a line  $L$  the conditions  $C_1\varphi_1 = C_2\varphi_2$ ,  $\psi_1 = \psi_2$ , with given constants  $C_1$  and  $C_2$ . The problem is solved for some particular shapes of  $L$ .

*A. Weinstein* (College Park, Md.).

**Arpiarjan, Noubar.** Orthogonalité sur des familles de courbes isogrammes et sur les domaines qu'elles décrivent. *C. R. Acad. Sci. Paris* 226, 1790–1791 (1948).

This paper extends the author's results obtained for the isograms  $|Z| = k$ ,  $0 \leq k < 1$ , of a Jordan curve represented by  $Z = F(z)$  on the circumference of the circle  $|Z| = 1$ , to isograms of an open curve  $C_0$  representable by  $Z = F(z)$  on the circle  $|Z| = 1$ . The following theorems are announced. (1) Every open curve  $C_0$  possesses many sequences  $\{f_n(z)\}$  orthogonal with the same weight-function not only on  $C_0$  but

also on its isograms  $C_k$ . (2) A sequence  $\{f_n(z)\}$  orthogonal with a weight-function  $w(z)$  on the isograms  $C_k$ ,  $k_0 \leq k \leq k_1$ , is also orthogonal with the weight-function  $w(z) \cdot |F'(z)|$  over the area of any domain  $D(k', k'')$  swept out by  $C_k$ , when  $k$  varies from  $k' \leq k_0$  to  $k'' \leq k_1$ . A necessary and sufficient condition for the orthogonality on the isograms  $C_k$ ,  $k_0 \leq k \leq k_1$ , of a sequence  $\{f_n(z)\}$  orthogonal with the weight-function  $w(z)$  on the curve  $C_0$  is formulated in terms of the sequence  $\{\varphi_n(x)\}$  orthogonal with the weight-function  $p(x)$  on the real interval  $(-\pi, \pi)$ , where the sequence  $\{\varphi_n(x)\}$  generates  $\{f_n(z)\}$ . (3) If the sequence  $\{\varphi_n(x-i \log k)\}$  is orthogonal on the interval  $(-\pi, \pi)$  with the weight-function  $p(x-i \log k)$  for all values of  $k$  between  $k_0$  and  $k_1$ , then the sequence  $\{f_n(z)\}$  generated by  $\{\varphi_n(x)\}$  and orthogonal on  $C_0$  is also orthogonal on the isograms  $C_k$  of  $C_0$  for  $k_0 \leq k \leq k_1$ .

E. Kogbelians (New York, N. Y.).

Arpiarian, Noubar. Orthogonalité sur des familles de courbes, et orthogonalité superficielle avec une infinité de poids différents. C. R. Acad. Sci. Paris 226, 1948-1950 (1948).

Results obtained by the author for the family of isograms of a curve [see the preceding review] are generalized for a one-parameter family of rectifiable curves  $C_u$  generated, when another parameter  $v$  varies from  $v_0(u)$  to  $v_1(u)$ , so that  $x=x(u; v)$ ,  $y=y(u; v)$  and  $u=u(x; y)$ ,  $v=v(x; y)$  define a mapping of planes  $z=x+iy$  and  $u+iv$  on each other. The main result is the following theorem. If the sequence  $\{f_n(z)\}$  is orthogonal with the weight-functions  $w(z, u)$  on a family of rectifiable curves  $C_u$ ,  $u_0 \leq u \leq u_1$ , it is also orthogonal over the area swept by these curves, the weight-function being  $H(u) \cdot w(z, u) \cdot |\text{grad } u|$ , where  $H(u)$  is an arbitrary positive function of the real variable  $u$ , which is supposed to be expressed in terms of  $x$  and  $y$  and is thus considered as a function of  $z$ .

The weight-function  $H(u) \cdot w(z, u) \cdot |\text{grad } u|$  in general is not the modulus of an analytic function of the complex variable  $z$ , but the author states that a weight-function  $P(z)$ , which is positive and which verifies the necessary and sufficient condition  $P \cdot \Delta P = |\text{grad } P|^2$ , is such a modulus. In the particular case of the isograms  $C_k$  corresponding to  $Z = F(z)$ , the author states a complement as follows. If the sequence  $\{f_n(z)\}$  orthogonal on  $C_k$  has as its weight-function  $w(z)$  the modulus of an analytic function, then the most general weight-function for any area swept by  $C_k$  is given by  $|F(z)|^\alpha \cdot |F'(z)| \cdot w(z)$ , provided that the weight-function is the modulus of an analytic function, where the exponent  $\alpha$  is an arbitrary real constant.

E. Kogbelians.

Vijayaraghavan, T. A power series that converges and diverges at everywhere dense sets of points on its circle of convergence. J. Indian Math. Soc. (N.S.) 11, 69-72 (1947).

The author considers the lacunary series  $\sum_n x^n/n$  with  $\nu_n = \mu_n$  for  $n$  odd and  $\nu_n = 2\mu_n$  for  $n$  even. For the choice  $\mu_n = 3^n$ , the set of points on the unit circle at which the series converges, and at which it diverges, are each everywhere dense. If  $\mu_n$  is chosen as  $1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)$ , then each of these sets is not only dense, but has the cardinal of the continuum; however, the divergence set has measure zero.

R. C. Buck (Providence, R. I.).

Wilson, R. A note on a theorem of Édrei. J. London Math. Soc. 22 (1947), 247-252 (1948).

Let  $f$  be a uniform function, regular at the origin, and having  $q$  essential singularities in the closed plane. Let

$f(z) = \sum_n c_n z^n$ , and from these coefficients form the Hankel determinants  $D_k^{(n)}$ . Let  $\rho_i$  be the order of the  $i$ th singularity. Then a theorem of Édrei [Compositio Math. 7, 20-88 (1939); these Rev. 1, 210] asserts that

$$\limsup |D_k^{(n)}|^{1/n^2 \log n} \leq \exp(-1/(\rho_1 + \rho_2 + \dots + \rho_q)).$$

The author gives an elementary proof of this theorem along the lines used in an earlier paper [Proc. London Math. Soc. (2) 39, 363-371 (1935)] to establish the result for  $q=1$ .

R. C. Buck (Providence, R. I.).

Macintyre, A. J., and Wilson, R. Associated integral functions and singular points of power series. J. London Math. Soc. 22 (1947), 298-304 (1948).

Let  $\psi(z) = \sum_n c_n z^n = G(1/(1-z))$ , where  $G$  is entire, of (finite) order  $\rho$  and finite type. A function  $F$  exists of order  $\rho/(1+\rho)$  such that  $F(n) = c_n$  for  $n = 1, 2, \dots$ . The authors show that the functions  $\limsup_{r \rightarrow \infty} r^{-\rho/(1+\rho)} \log |F(re^{i\varphi})|$  and  $\limsup_{r \rightarrow \infty} r^{-\rho} \log |G(re^{i\varphi})|$  reach their maxima for the same values of  $\varphi$ , that is, that the directions of strongest growth of  $F$  and  $G$  are the same. The proof uses previous work of Macintyre on the Laplace transform [Proc. London Math. Soc. (2) 45, 1-20 (1938)], known properties of the gamma function, and Hadamard multiplication of series.

R. C. Buck (Providence, R. I.).

Climescu, Al. C. Sur une matrice attachée à toute suite de nombres. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi, Iași] 3, 141-152 (1948).

With the Taylor series (1)  $a_0 + a_1 x + a_2 x^2 + \dots$  ( $a_1 = 1$ ) the author associates the infinite matrix  $P$  whose general element  $a_{ij}$  ( $i, j = 1, 2, \dots$ ) is the coefficient  $a_{ij}$  in the series (1), and he considers the conditions which must be satisfied by the series (1) in order that the matrix  $P$  may be of rank 1 or 2, respectively. The following two results are representative. For the matrix  $P$  to be of rank 1 it is necessary and sufficient that for every set of prime numbers  $p, q, \dots, s$  and every set of positive integers  $\alpha, \beta, \dots, \sigma$  the relation  $a_n = (a_p)^\alpha (a_q)^\beta \dots (a_s)^\sigma$  holds when  $n = p^\alpha q^\beta \dots s^\sigma$ . If the matrix  $P$  associated with the series (1) is of rank 1, the radius of convergence of the series (1) is at most unity. [For the sake of the validity of later statements, line 5 on p. 1 should read " $A = |a_{ij}| = |a_{i+j}|$  ( $i, j = 0, 1, \dots$ )".] On p. 4, the words "et réciproquement" should be deleted from line 13, and the words "nécessaire et" from the statement of theorem 3. The proof of theorem 5 on p. 5 becomes valid upon replacement of the number 2 by the least integer  $m$  ( $m > 1$ ) for which  $a_m \neq 0$ .]

G. Piranian.

\*Royo López, Jose. Metodos de Prolongación Analítica de las Series de Interpolación. [Methods of Analytic Continuation of Interpolation Series]. Instituto "Jorge Juan" de Matemáticas, Madrid, 1946. 59 pp.

The Dirichlet series  $\sum_n a_n n^{-z}$ , factorial series

$$\sum_n a_n n! / z(z+1)(z+2) \dots (z+n),$$

and Newton series

$$\sum_n (-1)^n a_n (z-1)(z-2) \dots (z-n)/n!$$

are said to be associated. The author first states a number of known results on the conditional and absolute convergence of such series and proves a theorem of Landau which asserts that associated series have the same region of convergence, except possibly for the points  $0, \pm 1, \pm 2, \dots$ . The notion

of overconvergence is introduced and several examples are given. Two theorems on the overconvergence of Newton series with gaps are stated and proved. One of these may also be obtained from the corresponding theorem for Dirichlet series by means of a more precise form of Landau's theorem, which the author formulates as follows: if one of the associated series has a subsequence of the sequence of partial sums which converges in a neighborhood of a point  $z_0$  of the line of convergence, then the corresponding subsequences for the other series also converge in the same neighborhood. The original Landau theorem is also generalized. With the series  $\sum a_n \exp(-\lambda_n z)$  are associated a factorial series  $\sum a_n \rho_1 \rho_2 \cdots \rho_n / (z + \rho_1)(z + \rho_2) \cdots (z + \rho_n)$  and a Newton series  $\sum (-1)^n a_n (z - \rho_1)(z - \rho_2) \cdots (z - \rho_n) / \rho_1 \rho_2 \cdots \rho_n$ , where  $\rho_n = 1/(\lambda_n - \lambda_{n-1})$ . Assume  $\sum 1/\rho_n$  divergent and  $\sum 1/|\rho_n|^2$  convergent. Then, excluding the points  $\pm \rho_n$ , these series converge for the same points  $z$ . Using this, together with known theorems on the convergence of generalized Dirichlet series, corresponding theorems are obtained for the generalized factorial and Newton series. Application to analytic extension by the method of rearrangement of series is indicated.

R. C. Buck (Providence, R. I.).

Schiffer, Menahem. **Faber polynomials in the theory of univalent functions.** Bull. Amer. Math. Soc. 54, 503-517 (1948).

With respect to (1)  $f(z) = z + c_0 + c_1 z^{-1} + c_2 z^{-2} + \cdots$ ,  $F_m(t)$  is the  $m$ th Faber polynomial if it is the unique polynomial in  $t$  of degree  $m$  such that  $F_m(f(z)) = z^m + \sum_{n=1}^m c_{mn} z^{-n}$ . The author establishes a generating function for all the Faber polynomials with respect to  $f(z) : \log(f(z) - t)/z = -\sum m^{-1} F_m(t) z^{-m}$ , and uses it to make a study of the variation of  $F_m(t)$  resulting from a variation in  $f(z)$ . If  $\zeta = f(z)$  is regular and univalent in a domain  $D$ , containing the point at infinity and bounded by a finite number of proper continua, it maps  $D$  upon a domain  $\Delta$  in the  $\zeta$ -plane. If  $\zeta_0$  is an arbitrary point of the  $\zeta$ -plane not belonging to  $\Delta$  and  $\rho$  a sufficiently small positive constant,  $|a| < 16$ , the function  $f^*(z) = f(z) + a\rho^2/(f(z) - \zeta_0) + \cdots$  is regular and univalent in  $D$  and maps  $D$  on a domain  $\Delta^*$  which is a small variation of the domain  $\Delta$  for small  $\rho$ . The resulting variations in  $F_m(t)$ ,  $c_{mn}$  are

$$F_m^*(t) = F_m(t) - a\rho^2 \cdot \frac{F'_m(t) - F'_m(\zeta_0)}{t - \zeta_0} + o(\rho^2),$$

$$F_m^*[f^*(z)] = F_m[f(z)] + a\rho^2 \left[ \frac{F'_m(\zeta_0)/f(z) - \zeta_0}{t - \zeta_0} \right] + o(\rho^2),$$

$$c_{mn}^* = c_{mn} + n^{-1} a\rho^2 F'_m(\zeta_0) F'_m(\zeta_0) + o(\rho^2).$$

The author applies these formulae to solve two extremum problems of the family  $\phi$  of all functions (1) univalent in  $D$ . Let  $\mathbf{x} = (x_1, \dots, x_N)$  denote a vector of  $N$  complex numbers, not all zero. The problem of determining the maximum modulus of the quadratic form  $Q(x, z) = \sum_{m,n=1}^N n c_{mn} x_m z_n$  and the functions  $f(z)$  of  $\phi$  for which the maximum is attained is solved. The extremal function  $f(z)$  maps the domain  $D$  upon a domain  $\Delta$  which is bounded by analytic slits  $\zeta(s)$  which satisfy  $\zeta'(s)^2 \cdot Q(x, z)^{-1} (\sum_{m=1}^N x_m F'_m[\zeta(s)])^2 = 1$ . Because of the complete square, the differential equation is integrable in closed form. Unique function pairs  $A_m(z)$ ,  $B_m(z)$ , regular in  $D$ , are associated with  $D$  so that, on each boundary continuum of  $D$ ,  $A_m(z) = \overline{B_m(z)} + \text{constant}$ ,  $A_m(z) = z^m + \sum_1^m a_{mn} z^{-n}$ ,  $B_m(z) = \sum_{n=1}^m b_{mn} z^{-n}$ . The inequalities, found by H. Grunsky [Math. Z. 45, 29-61 (1939)],

$$\left| \sum_{m,n=1}^N n(c_{mn} - a_{mn}) x_m z_n \right| \leq \sum_{m,n=1}^N n b_{mn} \bar{x}_m z_n$$

are obtained again by this variational method, together with the equation for the extremal functions:

$$\sum_{m=1}^N x_m F_m[f(z)] = \sum_{m=1}^N x_m A_m(z) + e^{i\gamma} \sum_{m=1}^N \bar{x}_m B_m(z),$$

$$e^{i\gamma} = \text{sgn} [Q(x, z) - \sum_{m,n=1}^N n a_{mn} x_m z_n].$$

A second extremal problem resulting from the functional

$$R(\omega, x) = \sum_{m,n=1}^N n c_{mn} x_m z_n + 2 \sum_{m=1}^N x_m F_m[f(\omega)] - \log f'(\omega)$$

again leads to a differential equation involving a perfect square term which the author is able to solve completely. In particular, the general inequalities obtained include the following inequality as a special case:

$$|\log f'(\omega) + \sum_{m,n=1}^N m^{-1} a_{mn} \omega^{-(m+n)}| \leq \sum_{m,n=1}^N n^{-1} b_{mn}(\omega)^{-n}(\omega)^{-m}.$$

M. S. Robertson (New Brunswick, N. J.).

Geronimus, Ya. L. **On certain extremal properties of analytic functions.** Izvestiya Akad. Nauk SSSR. Ser. Mat. 12, 325-336 (1948). (Russian)

The subscript  $k$  runs from 0 to  $s$ ;  $\{\alpha_k\}$  is a set of complex numbers with  $|\alpha_k| \leq r < 1$ ;  $A$  is the class of functions  $f(z)$  regular in  $|z| \leq 1$ ;  $M(f) = \max_{|z|=1} |f(z)|$ ,  $L(f) = \frac{1}{2} \pi^{-1} \int_0^{2\pi} |f(e^{i\theta})| d\theta$ ;  $f(z; \gamma)$  denotes a member of  $A$  satisfying  $f(\alpha_k) = \gamma_k$ ;  $\omega(f, c) = \sum c_k f(\alpha_k)$ ;  $B$  is the class of functions  $R(z) = R(z; c) = \sum c_k (z - \alpha_k)^{-1} + g(z)$ ,  $g \in A$ ;  $\Omega(R, \gamma) = \sum c_k \gamma_k$ . The following extremal problems are considered. To find

$$I_1 = \min M(f(z; \gamma)), \quad I_2 = \min L(f(z; \gamma)),$$

$$I_3 = \max |\Omega(R, \gamma)| / L(R), \quad I_4 = \max |\Omega(R, \gamma)| / M(R)$$

for given  $\gamma_k$ , where  $R$  ranges through  $B$ ,  $f \in A$ . Also to find

$$J_1 = \min M(R(z; c)), \quad J_2 = \min L(R(z; c)),$$

$$J_3 = \max |\omega(f, c)| / L(f), \quad J_4 = \max |\omega(f, c)| / M(f)$$

for given  $c_k$  where  $f$  ranges through  $A$ ,  $R \in B$ . The main result is that all extremal functions are rational functions whose form can be explicitly given and that  $I_1 = I_3$ ,  $I_2 = I_4$ ,  $J_1 = J_3$ ,  $J_2 = J_4$ .

If  $P(z) = \prod (z - \alpha_k)$ ,  $P^*(z) = \prod (1 - \bar{\alpha}_k z)$ , then we may put  $R(z) \in B$  into the form  $R(z) = (P^*(z) / P(z)) g(z)$ ,  $g \in A$ . Since  $|P^*/P| = 1$  on the unit circle,  $M(R) = M(g)$ ,  $L(R) = L(g)$ . Also  $\Omega(R, \gamma) = \omega(g, c)$ , if  $\gamma_k$  and  $c_k$  are connected by  $c_k = (P^*(\alpha_k) / P'(\alpha_k)) \gamma_k$ . The problems  $I_n$  and  $J_n$  ( $n = 1, \dots, 4$ ) are therefore equivalent. To prove  $I_1 \geq I_3$  it is sufficient to observe that

$$(1) \quad |\Omega(R, \gamma)| = \left| \frac{1}{2\pi i} \int_{|z|=1} R(z) f(z; \gamma) dz \right| \leq L(R) M(f(z; \gamma)).$$

Problem  $(I_1)$  was solved completely by Pick [Math. Ann. 77, 7-23 (1915)]. Its extremal function is a rational function  $f_1$  of constant modulus on  $|z| = 1$ . Putting  $f(z; \gamma) = f_1(z; \gamma)$  in (1) and investigating the conditions of equality in (1) leads to an essentially unique extremal function  $R$  for  $(I_1)$  for which  $I_1 = I_3$ . Because of the equivalence noted above this also solves the problems  $(J_1)$  and  $(J_3)$  and proves  $J_1 = J_3$ . Using a theorem due to Kakeya [Trans. Amer. Math. Soc. 22, 489-504 (1921)] it is shown that, for suitable  $c_k$ ,  $(J_4)$  has an extremal function  $f_2$  which is an  $f(z; \gamma)$ . Therefore, for these  $c_k$  and for any other  $f(z; \gamma)$ ,

$$|\omega(f, c)| / L(f) = |\sum c_k \gamma_k| / L(f) \leq |\omega(f_2, c)| / L(f_2) = |\sum c_k \gamma_k| / L(f_2)$$

and so  $f_2$  is an extremal function for  $I_2$ . Similarly the solution of  $(I_4)$  is obtained from the solution of  $(J_1)$  and  $I_2 = I_4$  follows from  $J_1 = J_4$ .

The case of several coincident  $\alpha$  is discussed. In particular, the classical results of Carathéodory and Fejér [Rend. Circ. Mat. Palermo 32, 218–239 (1911)] are obtainable from the author's work. The same is true of several previous results due to Kakeya, Golusin and others.

W. H. J. Fuchs (Ithaca, N. Y.).

Macintyre, Sheila Scott. An upper bound for the Whittaker constant  $W$ . J. London Math. Soc. 22 (1947), 305–311 (1948).

The number  $W$  is the smallest number for which the following theorem holds: if  $f$  is an entire function of exponential type 1 so that  $\limsup_{r \rightarrow \infty} r^{-1} \log M(r) \leq 1$ , and if  $f$  and each of its derivatives have at least one zero in  $|z| \leq \rho < W$ , then  $f$  must be identically zero. Previous results of Levinson [Duke Math. J. 11, 729–733 (1944); these Rev. 6, 122], and Boas [Duke Math. J. 11, 799 (1944); these Rev. 6, 123] show that  $.7199 < W < .7399$ . The present paper reports the sharper estimate  $W < .737756$ . The author uses the function  $f_n$  which is the solution of the equation  $f'(z) = f(e^{i\alpha}z)$ . The modulus of the zero of  $f_n$  nearest to the origin gives an upper bound for  $W$ . Boas's estimate results by the choice of  $\alpha$  as  $\frac{1}{4}\pi$ ; the author's improvement, by the choice of  $\alpha$  as 2.3911 (137°). This value is chosen as affording nearly the best estimate on  $W$  for all  $\alpha$  near to  $\frac{1}{4}\pi$  (135°). [The reviewer does not agree that this is therefore the best for all  $\alpha$ ; the conjecture, however, is reasonable.]

R. C. Buck (Providence, R. I.).

Gurin, L. S. On an interpolation problem. Mat. Sbornik N.S. 22(64), 425–438 (1948). (Russian)

Let  $a_1, \dots, a_n$  be given points in the complex plane, and with a given  $f(z)$  associate the polynomial  $P_{n,p}(z)$  of degree  $n(p+1)-1$  such that  $P_{n,p}^{(k)}(a_j) = f^{(k)}(a_j)$  for  $k=0, n, 2n, \dots, np$  and  $j=1, \dots, n$ . Then we have

$$P_{n,p}(z) = \sum_{j=0}^n \{c_{j1}(z)f^{(np)}(a_1) + \dots + c_{jn}(z)f^{(np)}(a_n)\},$$

where the  $c_{jk}(z)$  are polynomials generalizing the Lidstone polynomials (which arise for  $n=2, a_1=0, a_2=1$ ). The author discusses the convergence of  $P_{n,p}(z)$  to  $f(z)$ , giving sufficient conditions in the general case and necessary conditions when the  $a_k$  are the vertices of a regular  $n$ -gon. These generalize results of Schmidli [Über gewisse Interpolationsreihen, Zurich thesis, 1942; these Rev. 4, 39] and the reviewer [Duke Math. J. 10, 239–245 (1943); these Rev. 4, 271] for Lidstone series, and have a similar form. Entire functions of exponential type less than a certain  $\rho$  can be represented, those of larger type cannot; among functions of type  $\rho$ , finer distinctions must be made. When the  $a_k$  are the  $n$ th roots of unity,  $\rho$  is the absolute value of the root closest to 0 of  $\sum_{k=0}^n z^{kn}/(kn)!$ , and is asymptotically  $n/e$  for large  $n$ .

R. P. Boas, Jr. (Providence, R. I.).

Carlson, Fritz. Sur les fonctions entières. Ark. Mat. Astr. Fys. 35A, no. 14, 18 pp. (1948).

Démonstration de deux théorèmes sur les zéros des polynômes sections des fonctions entières énoncés en 1924 [C. R. Acad. Sci. Paris 179, 1583–1585 (1924)]. Le plus précis concerne le cas de l'ordre infini. Si  $f(z) = 1 + a_1 z + \dots + a_n z^n + \dots$  est la fonction entière d'ordre infini,  $P_m(z)$  le polynôme formé par les  $m+1$  premiers termes et  $\epsilon, \delta, \xi_0$  des nombres positifs

donnés arbitraires, il existe une suite infinie de valeurs  $m$  et de cercles correspondants,  $|z| = R_m$ , tels que le nombre des zéros de  $P_m(z)$  dans le secteur  $R_m(1-\delta) \leq |z| \leq R_m(1+\delta)$ ,  $|\arg z - \nu| \leq \xi$ ,  $\xi > \xi_0$ ,  $0 \leq \nu < 2\pi$ , soit égal à  $(\xi/\pi)m(1+\eta)$ ,  $|\eta| < \epsilon$ , pourvu que  $m$  soit assez grand ( $m > m_0(f, \epsilon, \delta, \xi_0)$ ). Dans le cas de l'ordre fini positif, l'auteur donne seulement une borne inférieure du nombre des zéros. Les démonstrations utilisent la régularisation des coefficients par la méthode de Newton-Hadamard, le calcul des résidus et un théorème de Carleman [même Ark. 15, no. 10 (1920)].

G. Valiron (Paris).

Doss, Shafik. Sur le comportement asymptotique des zéros de certaines fonctions d'approximation des séries de Dirichlet. Bull. Sci. Math. (2) 71, 165–179 (1947).

This paper is a continuation of a previous one [Ann. Sci. École Norm. Sup. (3) 64 (1947), 139–178 (1948); these Rev. 9, 422] devoted to zeros of approximating functions. In the present paper the author is concerned with partial sums or Riesz typical means of a Dirichlet series  $f(s) = \sum_{n=1}^{\infty} a_n e^{-ns}$ . If  $f(s)$  has a zero  $\zeta \neq 0$  of multiplicity  $k$  in the half-plane of convergence  $\Re(s) > \sigma_0$  and if  $\zeta_{n,j}$ ,  $j=1, 2, \dots, k$ , are the  $k$  zeros of the  $n$ th partial sum  $f_n(s)$  which converge to  $\zeta$  when  $n \rightarrow \infty$ , then  $\limsup_{n \rightarrow \infty} |\zeta_{n,j} - \zeta|^{1/(n+1)} = e^{\sigma_0 - \Re(\zeta)}$ . For the Riesz typical means of the first kind and integral order  $p$ , the author proves results which generalize his previous ones concerning Cesàro means. If  $k > p$ , the  $k$  roots of the  $n$ th mean  $f_{n,p}(s)$  which converge to a root  $\zeta$  of  $f(s)$  of multiplicity  $k$  located in the half-plane of convergence fall into two groups. The  $p$  zeros  $\zeta_{n,j}$  for which  $|\zeta_{n,j} - \zeta|$  is the largest satisfy  $\limsup_{n \rightarrow \infty} \lambda_{n+1}(\zeta_{n,j} - \zeta) = \alpha_j$ , where  $\alpha_j$  is the  $j$ th negative zero of the polynomial  $\sum_{n=0}^p \binom{n}{p} (\zeta)^n s^{p-n}$ . For the remaining  $k-p$  roots  $\limsup_{n \rightarrow \infty} |\zeta_{n,j} - \zeta|^{(k-p)/(n+1)} \leq e^{\sigma_0 - \Re(\zeta)}$  with equality for the case in which  $\lambda_{n+1}/\lambda_n \geq \eta > 1$ ,  $\eta$  fixed. If  $p \leq k$  only the first group is present.

E. Hille.

Minetti, Silvio. Sur l'allure des fonctions analytiques au voisinage d'une singularité essentielle. Pont. Acad. Sci. Acta 9, 169–186 (1945).

L'auteur considère les fonctions holomorphes dans un secteur  $0 < |z| < R$ ,  $|\arg z| < c$ , et étudie les valeurs limites lorsque  $z$  tend vers l'origine supposé point essentiel, ainsi que la distribution des points où une telle fonction  $f(z)$  prend une valeur donnée arbitraire. Dans ce dernier cas, il cherche à caractériser les directions de Picard, directions telles que dans tout voisinage angulaire de ces directions,  $f(z)$  prenne une infinité de fois toute valeur sauf au plus une valeur exceptionnelle. Les énoncés qu'il donne sans démonstrations paraissent être déduits de la théorie des familles normales de Montel. L'auteur semble croire [p. 171] que le fait pour une suite holomorphe  $f_n(z)$  de ne pas être normale en un point ou dans un domaine, faisait intervenir, avant ses travaux, des conditions superficielles, alors qu'il était connu et utilisé que, si les nombres  $f_n(z')$  sont bornés tandis que les nombres  $f_n(z'')$  ne le sont pas, la famille n'est pas normale dans tout domaine contenant  $z'$  et  $z''$ . Les premiers énoncés des parties III et IV du mémoire découlent de ce fait connu. Mais la proposition 7 [p. 182] considérée par l'auteur comme la plus avancée de ses recherches, semble douteuse au référent. Elle est mise en défaut par l'exemple de fonction entière donnée par Valiron [J. Math. Pures Appl. (9) 7, 113–136 (1928), p. 132] et plus simplement, pour les fonctions non entières, par les fonctions de pôles donnés, invariantes par la substitution  $(z, z)$  où  $s$  est réel supérieur à 1, considérées au voisinage du point à l'infini

dans un angle ne contenant ni l'origine, ni les pôles, mais contenant une infinité de zéros. *G. Valiron* (Paris).

**Milloux, H.** Le problème de la distribution des valeurs d'une fonction uniforme. *Mathematica, Timișoara* 23, 76-85 (1948).

Expository article.

**Heins, Maurice.** On the Denjoy-Carleman-Ahlfors theorem. *Ann. of Math.* (2) 49, 533-537 (1948).

D'après le théorème de Denjoy-Carleman-Ahlfors, une fonction entière  $f(z)$  qui admet  $n$  valeurs asymptotiques finies distinctes est telle que  $\log M(r) > Cr^{n/2}$ ,  $r > r_0$ ,  $C$  étant une constante et  $M(r) = \max |f(z)|$  pour  $|z| = r$ . L'auteur apporte un complément en montrant que, si, en outre, (1)  $\liminf_{r \rightarrow \infty} r^{-n/2} \log M(r) < \infty$ ,  $f(z)$  est d'ordre fini égal à  $n/2$ ; il considère comme plausible que, dans ces conditions,  $\log M(r)$  est asymptotiquement égal à  $cr^{n/2}$ ,  $c$  étant une constante. Dans sa démonstration, il utilise le théorème de déformation d'Ahlfors; il introduit les fonctions sousharmoniques et la notion de largeur d'une demi-bande équivalant à une certaine portion finie d'un domaine compris entre deux chemins sur lesquels  $f(z)$  est bornée sans l'être dans le domaine. Il présente ainsi le théorème de Denjoy-Carleman-Ahlfors et l'hypothèse (1) sous forme sousharmonique. Comme dans le cas classique, le résultat vaut dès qu'il existe  $n$  chemins d'indétermination finie non adjacents (au lieu de  $n$  valeurs asymptotiques distinctes); l'exemple de Denjoy,  $(\sin z^{n/2})^2$ , montre que de telles fonctions existent. *G. Valiron* (Paris).

**Baganas, Nicolas.** Quelques compléments sur la résolution de l'identité  $f_1^2(z) - f_2^2(z)R(z) = 1$ . *C. R. Acad. Sci. Paris* 226, 2116-2117 (1948).

The equation in the title appeared in the author's earlier note [same vol., 1064-1066 (1948); these Rev. 9, 415]. Here  $R$  is a given polynomial with no multiple zeros and the  $f_i$  are unknown integral functions of finite order. Three theorems are stated without proof. The first two discuss the relation of the orders of the  $f_i$  to the degree of  $R$ . The third determines those  $f_i$  which can be expressed by means of hyperelliptic integrals. *J. F. Ritt* (New York, N. Y.).

**Combes, Jean.** Sur un critère de normalité pour les familles de fonctions algébroïdes. *C. R. Acad. Sci. Paris* 227, 28-30 (1948).

The author announces results on entire algebroïd functions obtained by the method of normal families. These generalize recently announced results [same C. R. 226, 379-381 (1948); these Rev. 9, 341] pertaining to algebroïds of the form  $u = [E(z)]^{1/m}$ . *M. Heins* (Providence, R. I.).

**Stoilow, S.** Remarques sur la définition des points singuliers des fonctions analytiques multiformes. *Acad. Roum. Bull. Sect. Sci.* 26, 671-672 (1946).

These remarks concern the connection between the nature of the singular points of a multiple-valued function and the author's topological definition of a Riemann surface.

*L. Ahlfors* (Cambridge, Mass.).

**Pfluger, Albert.** Sur une propriété de l'application quasi conforme d'une surface de Riemann ouverte. *C. R. Acad. Sci. Paris* 227, 25-26 (1948).

It is proved that the property that a Riemann surface has a zero boundary in the sense of R. Nevanlinna is invariant under quasi-conformal mappings. *L. Ahlfors*.

**Ahlfors, Lars V.** Normalintegrale auf offenen Riemannschen Flächen. *Ann. Acad. Sci. Fenniae. Ser. A. I. Math.-Phys.* no. 35, 24 pp. (1947).

The author considers parabolic Riemann surfaces  $F$ , that is, those having no Green's functions. An Abelian integral (of the first kind) on  $F$  is a function which is regular at each point of  $F$  and which has a uniquely determined derivative at each point in term of the local uniformizing variable. It is assumed that the Dirichlet integral of the Abelian differential is finite. The assumptions that  $F$  is parabolic and that the Dirichlet integral is finite were made by R. Nevanlinna [same Ann. Ser. A. I. Math.-Phys. no. 1 (1941); these Rev. 7, 427], and the author states that his main purpose is to give a construction based on the  $A$ -periods. In Nevanlinna's treatment the classical normalization using  $A$ -periods was lost.

The first half of the paper is devoted to purely topological considerations. Let  $F_0 \subset F_1 \subset \dots \subset F_n \subset \dots$  be an increasing sequence of subdomains of  $F$  which exhaust the whole surface and let  $\Gamma_n$  be the boundary of  $F_n$ . It is shown that there is a canonical homology-basis  $A_1, B_1, A_2, B_2, \dots$  with the following two properties: (1) certain sections  $A_1, B_1, A_2, B_2, \dots, A_{k_n}, B_{k_n}$  are homology-bases for  $F_n \bmod \Gamma_n$ ; (2) cycles on  $\Gamma_n$  are homologous to linear combination of pure  $A$ -cycles whose indices tend to infinity with  $n$ .

The second half of the paper is mainly concerned with investigating the validity of the formal relation

$$D(u_1, u_2) = \sum_i \left( \int_{A_i} du_1 \int_{B_i} dv_2 - \int_{A_i} dv_2 \int_{B_i} du_1 \right),$$

where  $w_1 = u_1 + iv_1$ ,  $w_2 = u_2 + iv_2$  are two Abelian integrals. Let  $g_n$  be the function harmonic in  $F_n - F_0$  which is zero on  $\Gamma_0$  and 1 on  $\Gamma_n$ , and let  $h_n$  be its conjugate. A parabolic surface is characterized by the property that  $\lim g_n = 0$  as  $n$  tends to infinity. For a suitable method of exhaustion  $F_0 \subset F_1 \subset \dots \subset F_n \subset \dots$ , let  $\bar{F}$  denote the part of the surface bounded by the level curve  $\Gamma: g_n = t$ , and let  $g, h$  be the functions corresponding to  $\bar{F}$ . Denoting the boundary components of  $\bar{F}$  by  $\Gamma^i$ , let  $G_i$  be the subregion of  $\bar{F} - F_0$  which is swept out by the level curves  $h = \text{constant}$  which end on  $\Gamma^i$  and let  $\int_{\Gamma^i} dh = \Theta_i$ . Define functions  $\bar{u}_1, \bar{v}_2$  which in  $G_i$  are given by

$$\bar{u}_1 = u_1 - \frac{h}{\Theta_i} \int_{\Gamma^i} du_1, \quad \bar{v}_2 = v_2 - \frac{h}{\Theta_i} \int_{\Gamma^i} dv_2.$$

These functions are piecewise harmonic and determined up to additive constants. For a suitable limiting process it is shown that

$$D(u_1, u_2) = \lim \sum_i \left( \int_{A_i} d\bar{u}_1 \int_{B_i} d\bar{v}_2 - \int_{A_i} d\bar{v}_2 \int_{B_i} d\bar{u}_1 \right).$$

Here  $\bar{u}_1, \bar{v}_2$  and  $k$  depend upon  $n$  in a certain way. It then follows that if  $u_1$  or  $v_2$  has only finitely many periods different from zero or if  $u_1$  and  $v_2$  have only finitely many  $A$ -periods, we have

$$D(u_1, u_2) = \sum_i \left( \int_{A_i} du_1 \int_{B_i} dv_2 - \int_{A_i} dv_2 \int_{B_i} du_1 \right).$$

In the final section of the paper normal integrals are discussed. *D. C. Spencer* (Stanford University, Calif.).

**Nevanlinna, Rolf.** Über das Anwachsen des Dirichlet-integrals einer analytischen Funktion auf einer offenen Riemannschen Fläche. *Ann. Acad. Sci. Fenniae. Ser. A. I. Math.-Phys.* no. 45, 9 pp. (1948).

The author is concerned with harmonic functions on abstract Riemann surfaces and their Dirichlet integrals; in

particular, attention is focused on Riemann surfaces with null boundary and the behaviour of harmonic functions in the neighborhood of the boundary. The following lemma, which expresses an extremal property of harmonic measure, is the fundamental tool of the author. Let  $G$  denote a compact region of a Riemann surface  $F$ , the boundary of  $G$  being two disjoint closed sets  $\gamma_0$  and  $\gamma$ , each of which consists of a finite number of analytic arcs; let  $u$  be harmonic in  $G$ , vanish continuously on  $\gamma_0$  and satisfy  $\int_{\gamma_0} |\partial u / \partial n| ds = 1$ ; let  $x$  satisfy the same conditions as  $u$  and in addition assume continuously a constant value on  $\gamma$ . Let  $D(u, G)$  denote the Dirichlet integral of  $u$  on  $G$ . Then  $D(u, G) \leq D(x, G)$ . The proof is based on a study of  $D(u, G_k)$ , where  $G_k$  is the set  $[0 \leq |x| \leq \lambda (D(x, G))]$ .

A number of results follow with the aid of this lemma. For example, (1) if a function is harmonic (not identically constant) on a Riemann surface  $F$  with null boundary, its Dirichlet integral on the surface is infinite; (2) if  $G_0$  is a noncompact region of  $F$  (with null boundary), the relative boundary  $\gamma_0$  of  $G_0$  being compact and consisting of a finite number of analytic arcs, and if  $u$  is harmonic in  $G_0 + \gamma_0$ , then  $D(u, G_0) < +\infty$  if and only if  $u$  is bounded in  $G_0$ ; (3) if  $u$  is bounded in  $G_0$ , then  $u$  attains its maximum and minimum on  $\gamma_0$ .

M. Heins (Providence, R. I.).

Dalzell, D. P. Convergence of certain series associated with Fuchsian groups. *J. London Math. Soc.* 23, 19–22 (1948).

Let  $G$  denote a Fuchsian group properly discontinuous at some point of  $|z|=1$ ,  $T$ , a generic member of  $G$ ,  $a_s = T^{-s}$ . It is known that  $\sum a_s (1 - |a_s|^2) < +\infty$ . The author shows that, if  $G$  contains only hyperbolic transformations, then the series

$$\sum_a (1 - |a_s|^2) \{ \log 1/(1 - |a_s|^2) \}^k$$

converges,  $k$  being a nonnegative integer. The proof is based on appraisals of the integral

$$\int_{|z|=1} \sum_a |T'| \{ \log 1/(1 - |z|^2) \}^k dz$$

and induction on  $k$ .

M. Heins (Providence, R. I.).

Bernštein, S. N. On entire functions of finite degree of several complex variables. *Doklady Akad. Nauk SSSR (N.S.)* 60, 949–952 (1948). (Russian)

The author describes an entire function of two complex variables  $G(x, y) = \sum_{i,j=0}^{\infty} (a_{i,j}/k!l!) x^i y^j$  as of degree  $(p, q)$  if for arbitrary nonnegative  $\alpha, \beta$  ( $\alpha + \beta = 1$ ) we have  $\limsup |a_{i,j}|^{1/(k+l)} \leq p^q q^p$ , as  $k+l \rightarrow \infty$  with  $k/(k+l) \rightarrow \alpha$ ,  $l/(k+l) \rightarrow \beta$ ; the extension to more than two variables is obvious. A function  $f(x)$  of the single real variable  $x$  is said to be of finite growth with majorant  $H(x)$  if  $|f(x)| \leq H(x)$  on the whole real axis and  $H(x)$  is subject to the additional condition that, if  $f(x) = G_p(x)$ , an entire function degree  $p$  (i.e., of exponential type  $p$ ), this inequality implies  $|G_p(x)| \leq H_{k,p}(x)$ ,  $k = 1, 2, \dots$ , where  $\limsup_{k \rightarrow \infty} \{H_{k,p}(x)\}^{1/k} \leq p$ . The author states as lemmas two characterizations of such majorants  $H(x)$ . He calls  $f(x, y)$  of finite growth if  $|f(x, y)| \leq H(x)L(y)$ , where  $L(y)$  is a function of the same character as  $H(x)$ . His main theorem is then as follows. Let  $f(x, y)$  be of finite growth and of degree  $p$  in  $x$  for each real  $y$ , of degree  $q$  in  $y$  for each real  $x$ . Then  $f(x, y)$  is an entire function of degree  $(p, q)$  in the complex variables  $(x, y)$  and satisfies, for real  $(x, y)$ , the inequality  $|\partial^{k+l} f(x, y) / \partial x^k \partial y^l| \leq H_{k,p}(x)L_{l,q}(y)$ . Two corollaries are given. (1) If  $f(x_1, \dots, x_k)$  is bounded

by  $M$  in absolute value for real  $x_1, \dots, x_k$ , and if  $\limsup_{k \rightarrow \infty} |\partial^k f / \partial x_1^k| \leq p_1$  for each  $j$ , then for  $h_1 + \dots + h_k = m$  we have  $|\partial^m f / \partial x_1^{h_1} \dots \partial x_k^{h_k}| \leq M p_1^{h_1} \dots p_k^{h_k}$ . (2) If  $f(x_1, \dots, x_k)$  is of finite growth and of degree not exceeding  $p_i$  in each  $x_i$ , and we put  $x_i = \sum_{j=1}^k a_{j,i} t_j + b_i$ ,  $i \leq k$ , with real  $a_{j,i}$ , then  $f(x_1, \dots, x_k) = F(t_1, \dots, t_k)$  is an entire function of degree  $(P_1, \dots, P_k)$  in  $(t_1, \dots, t_k)$ , and  $P_i \leq \sum_{j=1}^k a_{j,i} p_j$ .

R. P. Boas, Jr. (Providence, R. I.).

\*Conforto, Fabio. *Funzioni Abeliane e Matrici di Riemann. Parte Prima*. Libreria dell'Università di Roma, 1942. 304 pp.

This volume is divided into two chapters, the first of which gives a self-contained development of the theory of Abelian functions, while the second relates the theory to algebraic geometry. The content of the first chapter is "self-contained" to the extent that it does not depend upon either the knowledge of the theory of Abelian integrals on an algebraic curve, or the theory of  $\theta$ -series; in effect, following a suggestion of Lefschetz [Bull. Nat. Res. Council 63 (1928), chap. 17, p. 354], it aims to build the theory of Abelian functions on the basis of nothing but a few theorems concerning the general theory of analytic functions of several variables, starting from the very definition of Abelian functions of  $p$  ( $\geq 1$ ) variables as functions (a) possessing  $2p$  independent periods, (b) which are proper functions of  $p$  variables, (c) meromorphic at finite distance. Hence this treatment differs in many ways from previous ones.

The first step is to prove that every Abelian function is the quotient of two intermediary functions. The author reaches this result by making use of the Cousin theorem, which states that every meromorphic function is the quotient of two integral transcendental functions (which vanish together only at the effective indetermination points of the function), and by proving that these functions can be so chosen as to be intermediary functions; a previous investigation of Appell [J. Math. Pures Appl. (4) 7, 157–219 (1891)] referring to the case  $p=2$  is partially taken into account in the proof of the last section. The author reduces the problem to a system of finite difference equations, which is solved by the aid of results of Hurwitz. Then, following Frobenius [J. Reine Angew. Math. 97, 16–48, 188–223 (1884)] and, to a certain extent, Castelnuovo, the author deduces, from the existence of intermediary functions, that there exists a principal form whose vanishing, together with the fulfillment of a few inequalities among the periods, gives the condition that the periods form a Riemann matrix.

Thus the condition that the period-matrix be a Riemann matrix is shown to be necessary in order that there exist Abelian functions with those periods. The sufficiency of the condition is proved by the author by retracing his steps and finding all intermediary functions belonging to a given principal form, hence all Abelian functions with given periods (as quotients of any two intermediary functions of the same type). In the course of the proof, starting from the Frobenius canonical expression for the principal form, the author reaches in a natural way the functional relations which are characteristic of the  $\theta$ -functions, and obtains their Fourier series. This leads ultimately to the well-known "existence theorem" for Abelian functions, and to the classification of all possible fields of Abelian functions. The first chapter ends with a critical comparison between the present development of the theory and previous ones.

In the second chapter the author defines, for any given field of Abelian functions of  $p$  variables, the related known

"Picard variety  $V_p$ ," i.e., an algebraic  $p$ -dimensional variety, defined up to a birational transformation, in general a one-to-one correspondence with the parallelopiped of periods, such that the functions of the field, and the independent variables, are respectively rational functions, and  $p$  independent simple integrals of the first kind on  $V_p$ . Making use of results of the first chapter, a simple proof of the Appell-Humbert theorem is given, stating that every algebraic  $(p-1)$ -dimensional subvariety of  $V_p$  (this being now supposed in a strictly one-to-one correspondence with the parallelopiped of periods) is fully represented by putting a suitable intermediary function equal to 0. The best known properties of a  $V_p$  with general moduli are then set forth (group of the birational transformations of  $V_p$  on itself; base, according to Severi, for the  $V_{p-1}$  of  $V_p$ , etc.); instances are given to show how such results must be modified in the case of special moduli. Finally the author proves the equivalence between the isomorphism (according to Scorza) of Riemann matrices and the rational correspondence of Picard varieties. The last part of the second chapter is intended to prepare for the "arithmetic" theory of Abelian functions (Scorza), which will be developed in a later volume.

E. Martinelli (Genoa).

**Mitchell, Josephine.** Value distribution of a meromorphic function of two complex variables on non-analytic manifolds. Duke Math. J. 15, 567-591 (1948).

Verf. schliesst sich an die Arbeiten von S. Bergman an, in denen er Funktionen von zwei komplexen Veränderlichen in Gebieten mit einer ausgezeichneten Randfläche untersucht [siehe Math. Ann. 104, 611-636 (1931); Jber. Deutsch. Math. Verein. 42, 238-252 (1933); Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] 1, 187-203 (1937); Mathematica, Cluj 14, 107-123 (1938)]. Der Rand  $m^2$  der untersuchten Bereich  $M^4$  soll aus zwei analytischen Hyperflächen bestehen, deren gemeinsame Punkte die ausgezeichnete Randfläche  $F^2$  bilden. Vom Maximum  $M$  der in  $M^4$  meromorphen Funktion  $f(z_1, z_2)$  wird auf die Werteverteilung in  $m^2 - F^2$  geschlossen. Insbesondere werden obere und untere Schranken für die Werte von  $|f|$  auf den zwei-dimensionalen Flächen  $K^2(a)$  gegeben, die in  $m^2 - F^2$  verlaufen und für jedes  $a$  einen Punkt auf der analytischen Fläche liefern, welche die  $a$ -te Faser von  $m^2 - F^2$  ist. In den weiteren Abschätzungen treten  $\max_{F^2} |f|$ ,  $\max_{K^2(a)} |f|$  und  $D(a, z_1, z_2)$  auf, wobei  $D(a, z_1, z_2) = \sum_j G[x_j, z_1; B^2(z_2^{(0)})]$  und  $G$  die Greensche Funktion des Schnittes  $B^2(z_2^{(0)})$ , der aus  $M^4$  durch  $z_2 = z_2^{(0)}$  gewonnen wird;  $(x_j, z_2^{(0)})$  durchläuft die  $a$ -Stellen von  $f(z_1, z_2^{(0)})$ . H. Behnke (Münster).

**Bergman, Stefan.** Functions of extended class in the theory of functions of several complex variables. Trans. Amer. Math. Soc. 63, 523-547 (1948).

Real- und Imaginärteil einer analytischen Funktion zweier komplexen Veränderlichen  $z_1$  und  $z_2$  ( $z_k = x_k + iy_k$ ,  $k = 1, 2$ ) sind biharmonische Funktionen, vom Verf. auch  $B$ -harmonische genannt. Sie genügen den Bedingungsgleichungen:

$$(1) \quad \partial^2 u / \partial x_k^2 + \partial^2 u / \partial y_k^2 = 0, \quad k = 1, 2; \\ (2) \quad \partial^2 u / \partial x_1 \partial x_2 + \partial^2 u / \partial y_1 \partial y_2 = 0, \quad \partial^2 u / \partial x_1 \partial y_2 - \partial^2 u / \partial x_2 \partial y_1 = 0.$$

Bekanntlich ist eine, z.B. in einem Dizylinder  $|z_1| < 1$ ,  $|z_2| < 1$ ,  $B$ -harmonische Funktion durch ihre Werte auf der nur zwei-dimensionalen "ausgezeichneten Randfläche"  $|z_1| = 1$ ,  $|z_2| = 1$  (Teil der drei-dimensionalen Randhyperfläche) eindeutig bestimmt. Gibt man jedoch umgekehrt

auf der ausgezeichneten Randfläche des Dizylinders stetige Randwerte vor, so existiert nicht immer eine zugehörige im Innern  $B$ -harmonische Funktion; wohl aber gibt es eine doppel-harmonische Funktion, welche die gegebenen Randwerte annimmt, d.h. eine Funktion, die den Gleichungen (1), aber nicht notwendigerweise den Gleichungen (2) genügt. Verf. stellt sich nun die Aufgabe, zu Bereichen des  $R_4$ , die von endlich vielen analytischen Hyperflächenstücken begrenzt werden, eine "erweiterte Klasse" ("extended class") von Funktionen so aufzustellen, dass bei Vorgabe von gleichmässig stetigen Randwerten auf der "ausgezeichneten Randfläche" (den Schnittflächen der Randhyperflächen) stets eine und nur eine zugehörige Funktion der Klasse im Innern des Bereiches existiert. Es wird dabei unter anderm verlangt, dass die "erweiterte" Funktionsklasse im Falle eines Dizylinders mit der Klasse der dort doppelharmonischen Funktionen identisch ist und ferner, falls eine  $B$ -harmonische Funktion existiert, die in allen Punkten der ausgezeichneten Randfläche grösser (kleiner) als die vorgegebenen Randwerte ist, dass sie dann auch im ganzen Innern des gegebenen Bereiches grösser (bzw. kleiner) als die Lösungsfunktion ist. Unter gewissen Zusatzbedingungen über die Bereiche und Randwerte gibt Verf. zwei Alternativen für die "erweiterte Klasse" an, welche das gestellte Problem lösen.

P. Thullen (Bogotá).

\*Vassal, Manoutcher. Sur Quelques Questions de Géométrie Intégrale des Espaces Hermitiens. Thesis, University of Geneva, 1940. 76 pp.

In the space of two complex variables  $(z_1, z_2)$ , consider the family of lines  $z_1 = az_2 + \beta$ , subject to the group of transformations  $Z_1 = az_1 + bz_2 + p$ ,  $Z_2 = a'z_1 + b'z_2 + p'$ , in which  $\begin{pmatrix} a & b \\ a' & b' \end{pmatrix}$  is an arbitrary unitary matrix, and  $p, p'$  arbitrary complex numbers. If we take the Haar measure in that group, then in the family of lines this introduces the group-invariant volume element  $(1 + a\bar{a})^{-1} da d\bar{a} db d\bar{b}$ . Now take a (nonanalytic) plane  $z_1 = \lambda + i\lambda' \sin \theta$ ,  $z_2 = \lambda' \cos \theta$ ,  $\lambda, \lambda'$  real parameters,  $\theta$  fixed, and in it an area of size  $\sigma$ . Then intersect the variable line with that area and "integrate" the corresponding number of intersections, either algebraically or absolutely. The resulting amount is then  $(2\pi \sin \theta)\sigma$  or  $\frac{1}{2}\pi(1 + \sin^2 \theta)\sigma$ . Thus it is maximal for  $\theta = \frac{1}{2}\pi$ , in which case the plane is likewise analytic. Similar results if the family of lines are themselves replaced by a group invariant family of (nonanalytic) planes, and also in case the underlying space, instead of being parabolic, is elliptic or hyperbolic.

S. Bochner (Princeton, N. J.).

**Wagner, Raphael D.** The generalized Laplace equations in a function theory for commutative algebras. Duke Math. J. 15, 455-461 (1948).

Une algèbre  $A$  de rang  $n$  sur le corps des réels  $R$  peut être considérée comme formée d'endomorphismes de l'espace vectoriel  $A$  (représentation régulière de  $A$ ); l'auteur dit qu'une application différentiable de  $A$  dans lui-même est analytique si en tout point la transformation linéaire tangente appartient à  $A$  (lorsque  $A$  est le corps des nombres complexes, cela redonne bien la définition usuelle). En supposant que l'algèbre  $A$  est une algèbre de Frobenius commutative et ayant un élément unité, l'auteur forme les "équations de Laplace généralisées" auxquelles doivent satisfaire les  $n$  composantes d'une fonction analytique.

J. Dieudonné (Nancy).

**Haefeli, Hans Georg.** *Hyperkomplexe Differentiale.* Comment. Math. Helv. 20, 382-420 (1947).

Der Verfasser hat sich die Aufgabe gestellt, die abbildungsgeometrischen Eigenschaften regulärer Quaternionenfunktionen und der entsprechenden Funktionen in weiteren hyperkomplexen Zahlbereichen zu untersuchen [vgl. Comment. Math. Helv. 17, 135-164 (1945); diese Rev. 7, 430]. Während in der eben erwähnten Arbeit dieses Problem für Quaternionenfunktionen behandelt wurde, betrifft die gegenwärtige die durch allgemeinere Funktionen vermittelten Abbildungen. Vorerst werden die regulären Funktionen in einem Linearsystem von Einheiten einer Cliffschen Algebra betrachtet. Die infinitesimalgeometrische Untersuchung der zugehörigen Abbildungen ergibt die in den Sätzen 1-6 ausgesprochenen Resultate. Die hierauf folgende Betrachtung der regulären Funktionen in einem Produktionsystem von Einheiten zweier Cliffscher Algebren ist deshalb von Wichtigkeit, weil unter diesen Funktionen die analytischen Funktionen mehrerer komplexer Variablen enthalten sind. Schliesslich werden Funktionen in Algebren, insbesondere Quaternionenfunktionen betrachtet, wobei vor allem auf die neuen, in den Sätzen 13-18 ausgesprochenen Resultate hingewiesen sei. Ferner enthält die Arbeit eine Literaturzusammenstellung, in welcher alle zu diesem Gebiete gehörigen Arbeiten zusammengestellt sind.

W. Nef (Fribourg).

### Theory of Series

**Alexiewicz, A.** *On multiplication of infinite series.* Studia Math. 10, 104-112 (1948).

Let  $N$  denote the set of all pairs  $(j, k)$  of positive integers, and let  $N_1, N_2, N_3, \dots$  be finite mutually exclusive subsets of  $N$  such that  $N = N_1 + N_2 + \dots$ . The terms of two series  $u_1 + u_2 + \dots$  and  $v_1 + v_2 + \dots$  then determine a product series  $w_1 + w_2 + \dots$  with terms defined by  $w_n = \sum u_i v_j$ , this sum being taken over all pairs  $(j, k)$  in  $N_n$ . The author characterizes the sequences  $N_n$  such that  $\sum w_n$  converges and  $\sum w_n = \sum u_n \sum v_n$  whenever  $\sum u_j$  and  $\sum v_k$  converge. Moreover, he characterizes those such that the equality holds whenever one of  $\sum u_j$  and  $\sum v_k$  converges and the other converges absolutely. For the case in which each subset  $N_n$  contains only one element, R. Rado [Quart. J. Math., Oxford Ser. 11, 229-242 (1940); these Rev. 2, 277] characterized the sequences  $N_n$  for which  $\sum w_n = \sum u_n \sum v_n$  whenever  $\sum u_j$  and  $\sum v_k$  converge. Unlike the proof of Rado, the proofs of Alexiewicz develop and use some basic results of a theory of functionals  $f(x, y)$ , defined for  $x$  in one Banach space and  $y$  in another, which are additive and continuous in each variable separately.

R. P. Agnew.

**Huzurbazar, V. S.** *Extensions of the limit theorems of Cauchy and Cesàro.* J. Univ. Bombay (N.S.) 16, Part 5, Sect. A, 1-10 (1948).

For each  $p=1, \dots, q$  let  $a_{p,1}, a_{p,2}, a_{p,3}, \dots$  be a sequence convergent to  $A_p$ . Then

$$\lim_{n \rightarrow \infty} \binom{n+q}{q}^{-1} \sum_{p_1 + \dots + p_q = n} a_{1,p_1} a_{2,p_2} \dots a_{q,p_q} = A_1 A_2 \dots A_q.$$

This generalizes the familiar fact that

$$\lim_{n \rightarrow \infty} n^{-1} (a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0) = AB$$

whenever  $a_n \rightarrow A$  and  $b_n \rightarrow B$ . The reciprocal of the coeffi-

cient of the  $\sum$  above is the number of terms in the sum, and the result is made intuitively obvious by the fact that the terms are bounded and, when  $n$  is large, most of the terms are near  $A_1 A_2 \dots A_q$ . It is proved in two ways that

$$\left( \sum_{p_1=0}^{\infty} a_{1,p_1} \right) \left( \sum_{p_2=0}^{\infty} a_{2,p_2} \right) \dots \left( \sum_{p_q=0}^{\infty} a_{q,p_q} \right) = \sum_{n=0}^{\infty} \sum_{p_1+\dots+p_q=n} a_{1,p_1} a_{2,p_2} \dots a_{q,p_q}$$

whenever all of the series involved are convergent. This generalizes a familiar theorem on Cauchy products. The simplest proof is an application of Abel's theorem on the continuity of power series. R. P. Agnew (Ithaca, N. Y.).

**Pettineo, B.** *Estensione di un teorema di Abel sulle serie numeriche reali.* Matematiche, Catania 1, 30-32 (1945).

The theorem that, if  $0 < u_n < M$  and  $\sum u_n = \infty$ , then  $\sum u_n / (u_1 + \dots + u_n)$  diverges as  $\log(u_1 + \dots + u_n)$ , is generalized. The hypothesis that  $u_n > 0$  is replaced by more general ones: that  $u_n$  is real and that the positive terms outweigh, in simple manners, the negative ones.

R. P. Agnew (Ithaca, N. Y.).

**\*Szász, Otto.** *Introduction to the Theory of Divergent Series.* Hafner Publishing Company, New York, N. Y., 1948. vi+72 pp. \$1.50.

This is a reprint of the author's lectures, published by the University of Cincinnati, 1944; these Rev. 6, 45.

**Lorentz, G. G.** *Eine Bemerkung über Limitierungsverfahren, die nicht schwächer als ein Cesàro-Verfahren sind.* Math. Z. 51, 85-91 (1948).

The author proves the following theorem, saying that the conditions are simpler than those of Mazur [Math. Z. 28, 599-611 (1928)]. Let  $0 < r \leq 1$ . A conservative method of summability defined by a transformation  $\sigma_n = \sum_{k=0}^{\infty} a_{n,k} S_k$  is not weaker than the Cesàro method  $C_r$  if and only if there is a constant  $M$  such that (1)  $\sum_{k=0}^{\infty} (k+1)^r |\Delta a_{n,k}| \leq M$ . If  $A$  is conservative and (1) holds, then  $A$  includes  $C_r$ . When  $r=1$ , the simpler condition was obtained by Orlicz [Tôhoku Math. J. 26, 233-237 (1926)]. The result fails when  $r > 1$ .

R. P. Agnew (Ithaca, N. Y.).

**Zygmund, A.** *On certain methods of summability associated with conjugate trigonometric series.* Studia Math. 10, 97-103 (1948).

By considerations involving conjugate trigonometric series, the author motivates the definition of two methods of summability whose properties resemble those of the classic methods  $(R, 1)$  and  $(R, 2)$  of Riemann. A series  $u_0 + u_1 + u_2 + \dots$  is summable  $(K, 1)$  to  $s$  if the series in

$$\sigma_1(\alpha) = u_0 + \sum_{n=1}^{\infty} (2u_n/\pi) \int_{-\pi}^{\pi} (\sin nt)/(2 \tan \frac{1}{2}t) dt$$

converges over some interval  $0 < \alpha < \alpha_0$  and  $\sigma_1(\alpha) \rightarrow s$  as  $\alpha \rightarrow 0+$ . This method, like  $(R, 1)$ , is not regular; but if  $\sum u_n$  converges to  $s$  then there is a subset  $E$  of the interval  $0 < \alpha < \alpha_0$  having right density 1 at the point 0, such that  $\sigma_1(\alpha) \rightarrow s$  as  $\alpha \rightarrow 0$  over  $E$ . If  $\sum u_n$  is summable  $(K, 1)$  to  $s$ , and satisfies the unilateral Tauberian condition  $nu_n > -C$ , then  $\sum u_n$  converges to  $s$ . A regular method  $(K, 2)$  is defined in terms of the transformation

$$\sigma_2(\alpha) = u_0 + \sum_{n=1}^{\infty} (2u_n/n\pi) \int_{-\pi}^{\pi} (\sin^2 \frac{1}{2}nt)/(2 \sin^2 \frac{1}{2}t) dt.$$

The following theorems are given with the proofs omitted because they resemble known proofs of corresponding theorems for  $(R, 1)$  and  $(R, 2)$ . If  $-1 < r < 0$ , then  $(K, 1)$  includes the Cesàro method  $C_r$ . If  $0 < r < 1$ , then  $(K, 2)$  includes  $C_r$ . If  $\sum u_n$  is summable  $C_1$  to  $s$ , then there is a set  $E$  of the type described above such that  $\sigma_2(a) \rightarrow s$  as  $a \rightarrow 0$  over  $E$ .

R. P. Agnew (Ithaca, N. Y.).

Timan, M. F. On the Abel summability of double series. *Doklady Akad. Nauk SSSR* (N.S.) 60, 1129-1132 (1948). (Russian)

Let  $\alpha, \beta > -1$ . A double series  $\sum u_{jk}$  is restrictedly summable by the Euler-Abel power series method if it is summable by the Cesàro method  $C_{\alpha, \beta}$  and its  $C_{\alpha, \beta}$  transform  $\sigma_{mn}$  is  $o(m^{\beta+1})$  for each  $n$  and is  $o(n^{\alpha+1})$  for each  $m$ . This is related to work of Ogieveckil [same *Doklady* (N.S.) 58, 1897-1900 (1947); these *Rev.* 9, 278] and to work of J. C. Vignaux and others quoted in the cited review.

R. P. Agnew (Ithaca, N. Y.).

Delange, Hubert. Quelques théorèmes taubériens. *C. R. Acad. Sci. Paris* 226, 1787-1790 (1948).

Tauberian theorems are proved for the transformation  $\Phi(x) = \int_0^\infty \phi(t, x) ds(t)$ ; the conclusions have the form  $\limsup |\Phi(ax) - s(x)| \leq W\tau(a)$ , where  $\tau(a)$  depends on the kernel  $\phi(t, x)$  and  $W$  on the Tauberian conditions satisfied by  $s(t)$ .

H. R. Pitt (Belfast).

Atkinson, F. V. Über die Stirlingsche Reihe. *Comment. Math. Helv.* 21, 332-335 (1948).

Stellt  $F(z)$  in einem Gebiet eine analytische Funktion dar und konvergiert die Reihe  $\sum_{n=0}^{\infty} \gamma_n F^{(n)}(z) = \varphi(z)$  (mit  $\gamma_0 = 1$ ) dort gleichmässig, so bekommt man

$$\sum_{r=0}^m \alpha_r \varphi^{(r)}(z) = \sum_{r=0}^m \alpha_r \sum_{n=0}^{\infty} \gamma_n F^{(n+r)}(z).$$

Werden die Zahlen  $\alpha_0, \dots, \alpha_m$  so gewählt, dass  $\alpha_0 = 1$  und  $\sum_{r=0}^h \alpha_r = 0$  für  $h > 0$ , so findet man

$$F(z) = \sum_{r=0}^m \alpha_r \varphi^{(r)}(z) + R_m,$$

wo  $R_m = \sum_{n=0}^{\infty} F^{(n)}(z) \sum_{h < r \leq m} \gamma_{h-r} \alpha_r$ . Diese Methode, auf  $P(z) = \log \Gamma(z)$  mit  $\gamma_n = 1/(n+1)!$  und  $\varphi(z) = z \log z - z + C$  angewandt, liefert die Stirlingsche Reihe.

J. G. van der Corput (Amsterdam).

\*Wall, H. S. *Analytic Theory of Continued Fractions*. D. Van Nostrand Company, Inc., New York, N. Y., 1948. xiii + 433 pp. \$6.50.

As indicated in the title, the author restricts himself to the study of continued fractions with complex terms from the viewpoint of analytic function theory; no attempt is made to treat the arithmetic aspects of continued fractions or their connections with the theory of numbers. This, together with the large body of new results obtained in recent years, permits the overlap of the present book with that of Perron [Die Lehre von den Kettenbrüchen, 2d ed., Teubner, Leipzig, 1929] to be relatively slight. Generally speaking, only the work of the author and his students is fully represented; however, the bibliography and references to recent literature are more than adequate, the former filling nine pages. Following each chapter is a set of exercises covering special cases and suggesting related problems not studied in the text, but to which references in the bibliography are given. The unwieldy but unambiguous

explicit displayed notation is used for all continued fractions. A summary follows. [The abbreviation "c.f." will be used for "continued fraction."]

The introduction contains a brief sketch of the historical development of the theory of continued fractions and an outline of the material covered by the book. In chapters I and II, the usual recurrence formulae for the numerator and denominator of an approximant (convergent) of a c.f., the determinant theorem, and certain equivalence operations are obtained; equivalent series are constructed and convergence of a c.f. is defined. An example is given to show that a convergent c.f. can be constructed having a subsequence of the partial numerators assigned almost at will. The divergence theorem of von Koch is proved and generalized. The chapter concludes with a treatment of periodic c.f.'s from the viewpoint of linear fractional transformations. In chapter III, convergence theorems are derived from the consideration of the equivalent series, among them, Worpitzky's theorem whereby  $K_1^*(a_n/1)$  converges if  $|a_n| \leq \frac{1}{2}$ ; generalizations, including the theorems of Pringsheim-Perron and Van Vleck, are proved, as is the convergence theorem of von Koch. The parabola theorem of Scott and Wall is also treated.

The definition of a positive definite c.f. is given in chapter IV, and general properties discussed. Sections 19 and 20 are devoted to a study of "chain sequences." The author terms a sequence  $\{x_n\}$  a chain sequence if there are numbers  $g_n$ ,  $0 \leq g_n \leq 1$ , such that  $x_n = g_n(1 - g_{n-1})$ ; these enter naturally into the study of c.f.'s. Chapter V is devoted to the "theorem of invariability," obtained by Hellinger and the author and generalizing an earlier result of Hamburger, together with applications to  $J$ -fractions. The Stieltjes type c.f. is treated in chapter VI by means of a related  $J$ -fraction. Perron's theorem on the existence of local convergence regions is proved in chapter VII, and extended by means of the parabola theorem.

Chapter VIII discusses the "value region problem." A pair  $(E, V)$  of sets is required such that if  $c_n \in E$  for all  $n$ , then every approximant of  $K_1^*(c_n/1)$  lies in  $V$ . General classes of such sets are found, all of circular or parabolic type, by means of linear mappings. Chapters IX and XI deal with various algorithms for obtaining the  $J$ -fraction expansion of rational functions and of power series. In chapter X, application is made to the theory of equations; the zeros of the polynomial  $P(x)$  are located by means of the expansion of  $N(x)/P(x)$ , where  $N(x)$  may be chosen as the alternant for  $P(x)$ .

In chapter XII, a brief exposition of portions of the Hellinger-Toeplitz theory of infinite matrices is given. Application to the study of  $J$ -fractions is made. In chapter XVI, the usual definitions of Hausdorff matrices are given, and conditions for the regularity of the corresponding summability methods are discussed. An example is given to show the connection of the inclusion problem with c.f. theory. In chapters XIII and XV, the representability of certain classes of analytic functions as Stieltjes transforms, or more general integral representations, is discussed in terms of c.f.'s, and general expansion theorems are obtained. In chapters XIV and XVII the moment problems are discussed in the framework of c.f. theory. Nevanlinna's solution of the Hamburger problem is simplified by consideration of asymptotic expansion in half planes only. Carleman's determinancy theorem for the Stieltjes problem is derived algebraically. The c.f. expansion of certain special functions is discussed in chapters XVIII and XIX. Several lengthy

tables are included. The book concludes in chapter XX with the definition of the Padé table for a power series and the location among its entries of c.f. approximants. Several conjectures are made as to properties and possible usefulness of the table. The  $C$ -fractions are defined, and used to expand arbitrary power series.

R. C. Buck.

Wall, H. S. *On some criteria of Carleman for the complete convergence of a  $J$ -fraction*. Bull. Amer. Math. Soc. 54, 528-532 (1948).

This paper contains a generalization to complex numbers  $a_p, b_p, p=1, 2, \dots$ , and  $b_p \neq 0$  in general, of the criterion of T. Carleman [Sur les Équations Intégrales Singulières à Noyau Réel et Symétrique, Uppsala, 1923; Les Fonctions Quasi Analytiques, Paris, 1926] for the complete convergence of a real  $J$ -fraction

$$-K \sum_{p=1}^{\infty} \frac{-a_{p-1}}{b_p + z}, \quad a_0 = 1, \quad a_p \neq 0.$$

E. Frank (Chicago, Ill.).

### Fourier Series and Generalizations, Integral Transforms

\*Ahiezer, N. I. *Lekcii po Teorii Approksimacii. [Lectures on the Theory of Approximation]*. OGIZ, Moscow-Leningrad, 1947. 323 pp. (Russian)

This is an interesting and valuable book presenting various aspects of the theory of approximation of functions in the real domain. Chapter I deals with problems of approximation in normed linear spaces, in particular in Hilbert space. It contains, among others, a proof of the theorem of Müntz on the completeness of the system  $x^n, x^{n+1}, \dots$ ; of F. Riesz's theorem on the general form of the functional in a complete (but not necessarily separable) Hilbert space; and of uniqueness of best approximation in strongly normed linear spaces. Chapter II is devoted to the Chebyshev approach to best approximation and gives properties of polynomials, and rational functions, of best approximation. Use is made of Haar's theorem on necessary and sufficient conditions for the uniqueness of approximation of continuous functions by linear combinations of functions from a given family. Uniqueness of best approximation in the spaces  $C$  and  $L$ , which are not strongly normed, is settled. Chapter III gives an introduction to harmonic analysis and contains, in particular, the inversion theorems of Plancherel and Watson. Chapter IV deals with extremal properties of entire functions of exponential type. The Paley-Wiener theorem about functions of exponential type and of class  $L^2$  is established by means of Borel's transformation. S. Bernstein's inequalities for the derivatives of functions of exponential type and bounded on the real axis (which contain as special cases his well-known inequalities for the derivatives of trigonometric polynomials) are presented in a still more general form involving the "conjugate" functions. Levitan's polynomials find several applications, in particular in the proof of an extension of the Fejér-Riesz theorem on positive trigonometric polynomials to functions of exponential type positive on the real axis. Chapter V deals with problems originating from the theorems of D. Jackson and S. Bern-

stein connecting the differential properties of functions with best approximation by trigonometric polynomials, and again these results are extended to entire functions of exponential type. The results of Favard, Ahiezer and Krein concerning the best inequalities for approximation are also extended to that case. Chapter VI centers around Wiener's general Tauberian theorem (with applications to Ikebara's and Carleman's theorems), the proof of which is presented in the form utilizing the notions of normed ring and of ideal. Finally an appendix contains miscellaneous complements and problems. The presentation is clear and elegant.

A. Zygmund (Chicago, Ill.).

Sunouchi, Gen-ichirō. *Notes on Fourier analysis. I. On the convergence test of Fourier series*. Math. Japonicae 1, 41-44 (1948).

The Hardy-Littlewood test for the convergence of a Fourier series [Ann. Scuola Norm. Super. Pisa (2) 3, 43-62 (1934)] is known to hold under two assumptions, one of which is a simple continuity restriction on the function in question, while the other is a two-sided Tauberian condition on the Fourier coefficients. This test has been extended by G. W. Morgan [Ann. Scuola Norm. Super. Pisa (2) 4, 373-382 (1935)] so as to permit more general types of behaviour both of the function and of its Fourier coefficients. Morgan's proof is very similar to that of Hardy and Littlewood, and is based on Valiron summation. F. T. Wang, on the other hand, gave a proof, using Riesz summation, of the original result of Hardy and Littlewood with a one-sided Tauberian condition [Proc. London Math. Soc. (2) 47, 308-325 (1942); these Rev. 4, 37]. This paper contains a theorem which combines the results of Morgan and Wang, and the proof is similar to theirs. K. Chandrasekharan.

Yano, Shigeki. *Notes on Fourier analysis. II. Proof of maximal theorems for Fourier series*. Math. Japonicae 1, 45-48 (1948).

[Cf. the preceding review.] A new proof of the theorem that, if  $s_n(x)$  is the  $n$ th partial sum of the Fourier series of an  $f \in L^2$ , then

$$\int_0^{2\pi} \max_n \frac{s_n(x)}{\log(n+2)} dx \leq A \int_0^{2\pi} f^2(x) dx.$$

[See also Hardy and Littlewood [Proc. Cambridge Philos. Soc. 40, 103-107 (1944); these Rev. 6, 47] where a proof of an equivalent inequality is given.]

A. Zygmund.

Lorentz, G. G. *Fourier-Koeffizienten und Funktionenklassen*. Math. Z. 51, 135-149 (1948).

Several inequalities involving the Fourier coefficients are obtained which give sufficient or necessary conditions that a function be in one of the more common function classes, i.e.,  $\text{Lip } \alpha$ ,  $\text{Lip } (\alpha, p)$ ,  $BV$ ,  $L^p$ . The inequalities are couched in the language of linear transformations, where the spaces mentioned have their usual norms. Let  $f(t)$  be integrable of period  $2\pi$  and let  $\{y\} = [a_1, b_1, a_2, \dots]$  be the sequence of Fourier coefficients of  $f(t)$ . (All functions are normalized by the condition  $a_0 = 0$ .) Let  $M(\alpha, p)$  and  $R(\alpha, p)$ ,  $0 < \alpha \leq 1$ ,  $1 \leq p \leq \infty$  be the spaces of sequences  $\{y\}$  such that  $\sup_n n^{-\alpha} \{ \sum_{k=1}^n (|a_k|^p + |b_k|^p) \}^{1/p} < \infty$  or  $\sup_n n^\alpha \{ \sum_{k=1}^n (|a_k|^p + |b_k|^p) \}^{1/p} < \infty$ , respectively, with the latter expressions as the norms in the respective spaces, and with the usual convention for  $p = \infty$ . If  $T$  denotes the transformation from function to series or vice versa, then

specimen results for  $\text{Lip } \alpha$  and  $BV$  state that  $T$  is a linear operation as follows:

$$\begin{aligned} T(\text{Lip } \alpha) &\subset R(\alpha + \frac{1}{2} - p^{-1}, p), & \alpha > p^{-1} - \frac{1}{2}; \\ T(\text{Lip } \alpha) &\subset M(p^{-1} - \alpha - \frac{1}{2}, p), & \alpha < p^{-1} - \frac{1}{2}; \\ T(R(\alpha, p)) &\subset \text{Lip } (\alpha - 1/p'), & 1/p' < \alpha < 1 + 1/p'; \\ T(BV) &\subset R(1/p', p), & 1 < p \leq \infty; \\ T(R(1, p)) &\subset BV, & 1 \leq p \leq 2; \\ T(R(\frac{1}{2} + 1/p', p)) &\subset BV, & 2 \leq p \leq \infty. \end{aligned}$$

For series with monotonic null-sequences of Fourier coefficients and for lacunary series, necessary and sufficient conditions are deduced that the functions be in  $\text{Lip } \alpha$ . In all cases it is shown that the results obtained are the best possible of their types. *P. Civin* (Eugene, Ore.).

**Salem, R.** Rectifications to the papers: Sets of uniqueness and sets of multiplicity, I and II. *Trans. Amer. Math. Soc.* 63, 595–598 (1948).

In the papers cited in the title [same Trans. 54, 218–228 (1943); 56, 32–49 (1944); these Rev. 5, 3; 6, 47] the author gives the theorem that a necessary and sufficient condition for a perfect set  $P$  of constant ratio  $\xi$  to be a set of uniqueness for trigonometric series is that  $\theta = 1/\xi$  be an algebraic integer all of whose conjugates are inside the unit circle. The author has now found that the proof of the sufficiency of the condition is not conclusive and that this part of the problem remains open. [The proof of the necessity is not affected.] Some new partial results are obtained for the sufficiency condition from which it follows in particular that the old result remains valid if  $\theta = 1/\xi$  is a quadratic integer whose conjugate is inside the unit circle.

*A. Zygmund* (Chicago, Ill.).

**Schmetterer, Leopold.** Zum Konvergenzverhalten gewisser trigonometrischer Reihen. *Monatsh. Math.* 52, 162–178 (1948).

The author considers the convergence of trigonometric series  $(*) \sum_{k=1}^{\infty} a_k \cos kx$  under the hypothesis that  $|a_k| \downarrow 0$ , and under various hypotheses concerning the behavior of  $\{\text{sgn } a_k\}$ . A sample result is the following. If the series  $(*)$  has alternating blocks of  $p$  positive and  $q$  negative coefficients ( $p, q > 0$ ) and if  $\sum_{k=1}^{\infty} |a_k|$  is divergent, then for all points of  $[0, 2\pi]$  not of the form  $(**) 2k\pi/(p+q)$ ,  $k = 0, 1, \dots, (p+q-1)$  the series converges, uniformly in every closed interval not containing a point  $(**)$ . Detailed statements are also established concerning the behavior at the points  $(**)$ . Corresponding investigations are made for the series  $\sum_{k=1}^{\infty} b_k \sin kx$  under the hypothesis  $|b_k| \downarrow 0$ . *P. Civin*.

**Singh, U. N.** On the strong summability of a Fourier series and its conjugate series. *Proc. Nat. Inst. Sci. India* 13, 319–325 (1947).

Let  $\tilde{s}_n(x)$  denote the partial sums of the series conjugate to the Fourier series of a function  $f$ . Let

$$\psi(t) = \psi(t, x) = \frac{1}{2} \{f(x+t) - f(x-t)\}.$$

If the integral  $\tilde{f}(x) = -\pi^{-1} \int_0^{\pi} \psi(t) \cot \frac{1}{2}t dt$  converges, and if  $\int_0^t |\psi(u)| du = O(t |\log t|^{-\alpha})$ , for some  $\alpha > \frac{1}{2}$ , then  $\sum_n [\tilde{s}_n(x) - \tilde{f}(x)]^2 = o(n)$ . [See also Wang, J. *London Math. Soc.* 19, 208–209 (1944); these Rev. 7, 246.]

*A. Zygmund* (Chicago, Ill.).

**Šneider, A. A.** On series of Walsh functions with monotonic coefficients. *Izvestiya Akad. Nauk SSSR. Ser. Mat.* 12, 179–192 (1948). (Russian)

The orthogonal system of Rademacher was first completed by Walsh [Amer. J. Math. 45, 5–24 (1923)]. Two different definitions of the Walsh functions were given independently by Kaczmarz [Comptes Rendus du I Congrès des Mathématiciens des Pays Slaves 1929, Warsaw, 1930, pp. 189–192; see also Kaczmarz and Steinhaus, *Theorie der Orthogonalreihen*, Warszawa-Lwów, 1935] and Paley [Proc. London Math. Soc. (2) 34, 241–264, 265–279 (1932)]. Both definitions lead to the same system but the orders of the functions within the systems are different. Let us denote the Paley and the Kaczmarz arrangements by  $\{\varphi_n\}$  and  $\{\psi_n\}$ , respectively, and let  $c_n$  be a monotonic decreasing sequence of positive numbers. The author shows that (1) the series  $\sum c_n \varphi_n(t)$  converges uniformly in every interval  $(\delta, 1-\delta)$ ,  $0 < \delta < 1$ ; (2) if  $c_n \not\asymp o(1/\log n)$ , then  $\sum c_n \psi_n(t)$  diverges almost everywhere; (3) if  $\sum c_n < \infty$ , then  $\sum c_n \psi_n(t)$  converges almost everywhere.

*A. Zygmund*.

**Krein, S. G., and Levin, B. Ya.** On the convergence of singular integrals. *Doklady Akad. Nauk SSSR (N.S.)* 60, 13–16 (1948). (Russian)

A singular integral is  $\int_a^b \varphi_n(x, t) f(t) dt$ , where the kernels satisfy  $(*) \lim_{n \rightarrow \infty} \int_a^b \varphi_n(x, t) dt = 1$  for  $a \leq x < t \leq b$ . Generalizing the classical problem of representation by singular integrals, the authors seek necessary and sufficient conditions on the  $\varphi_n$  so that for all functions  $f(x)$  of a specified class  $(**)$   $\lim_{n \rightarrow \infty} \int_a^b \varphi_n(x, t) f(t) dt = f(x)$  will hold at all points  $x$  for which  $\lim_{n \rightarrow \infty} \int_a^b \Theta_r(x, t) f(t) dt = f(x)$ ,  $0 < r < 1$  being a given one-parameter family of kernels. This problem is treated in a very general setting using Banach space methods. The following is one of the theorems obtained by specializing the general results. Let  $\varphi_n$ ,  $n \geq 1$ , be such that  $\int_a^b \varphi_n(t) f(t) dt$  exists for every  $f(t) \in L^p$  which at the point  $t=x$  is the derivative of its indefinite integral. Then necessary and sufficient conditions that  $(**)$  hold for all  $f(t) \in L^p$ ,  $p \geq 1$ , at points  $t=x$  where  $f(t)$  is the derivative of its indefinite integral are  $(*)$  and  $\|\varphi_n(x, t)\| \leq M(x)$  with  $M$  independent of  $n$ . Here  $\|\varphi\| = \min(\max(\|x\|_{L^q}, V[\psi]))$ , where  $q = p/(1-p)$ ,

$$V[\psi] = \int_a^x \varphi \psi(t) du + \int_x^b \varphi \psi(t) du$$

and  $\min$  refers to all decompositions  $\varphi(t) = \psi(t) + \chi(t)$  with  $\chi(t) \in L^q$  (there exist such with finite  $V[\psi]$ ). This theorem extends and completes results of Lebesgue [Ann. Fac. Sci. Univ. Toulouse (3) 1, 25–117, 119–128 (1909)] and P. Romanovski [Math. Z. 34, 35–49 (1931)] for the case  $p=1$ . *A. Doetsky* (Princeton, N. J.).

**Krein, S. G., and Levin, B. Ya.** On the strong representation of functions by singular integrals. *Doklady Akad. Nauk SSSR (N.S.)* 60, 195–198 (1948). (Russian)

[Cf. the preceding review.] The function  $f(x)$  is said to be strongly represented at the point  $x$  by the singular integral with kernels  $\varphi_n$  if  $(\dagger) \lim_{n \rightarrow \infty} \int_a^b \varphi_n(x, t) |f(t) - f(x)| dt = 0$ . The problem of this paper is to obtain necessary and sufficient conditions on the  $\varphi_n$  so that  $(\dagger)$  will be implied by  $\lim_{n \rightarrow \infty} \int_a^b \Theta_r(x, t) |f(t) - f(x)| dt = 0$ , where  $\Theta_r$ ,  $(0 < r < 1)$  is a given one-parameter family of nonnegative kernels. Using the methods of the paper reviewed above and results of M. Krein [C. R. (Doklady) Acad. Sci. URSS (N.S.) 28, 13–17 (1940); these Rev. 2, 315] on cones in Banach space, the authors obtain several results on the type of functional

they are interested in, including necessary and sufficient conditions for the weak convergence to zero of certain sequences of linear functionals. Their general theorems contain as a special case the results of D. Faddeev [Rec. Math. [Mat. Sbornik] N.S. 1(43), 351-368 (1936)] and B. I. Korenblum [same Doklady 58, 973-976 (1947); these Rev. 9, 347] on the representation by singular integrals of functions of  $L^p$ ,  $p \geq 1$ , at their Lebesgue points.

A. Dvoretzky (Princeton, N. J.).

Conzelmann, Rolf. Beiträge zur Theorie der singulären Integrale bei Funktionen von mehreren Variablen. II. Comment. Math. Helv. 21, 270-301 (1948).

Results obtained in the first part of this paper [same Comment. 19, 279-315 (1947); these Rev. 8, 458] are applied to classical kernels of several variables, in particular to kernels which are products of kernels depending on one variable only, and of standard analytic structure.

A. Zygmund (Chicago, Ill.).

Kolmogorov, A. N. A remark on the polynomials of P. L. Čebyšev deviating the least from a given function. Uspehi Matem. Nauk (N.S.) 3, no. 1(23), 216-221 (1948). (Russian)

Haar's necessary and sufficient conditions for the uniqueness of the "polynomial in  $f_k(x)$ ,"  $P_n(x) = c_1 f_1(x) + \dots + c_n f_n(x)$ , deviating the least from a given function  $F(x)$  of a real variable  $x$ , are extended to a complex variable.

E. Kogbellantz (New York, N. Y.).

Hardy, G. H. A double integral. J. London Math. Soc. 22 (1947), 242-247 (1948).

Une intégrale  $\int_{-\infty}^{\infty} h(x)dx$  est dite convergente v.p. si  $\int_{-\infty}^{\infty} = \lim_{x \rightarrow \infty} \int_x^{\infty}$ ; et convergente (C, 1) si

$$\int_{-\infty}^{\infty} = \lim_{x \rightarrow \infty} \int_x^{\infty} (1 - |x|/x) h(x) dx.$$

D'autre part si  $f(x) \in L^2$ , la transformée de Fourier de  $f(x)$  s'écrit  $F(\lambda) = \lim_{x \rightarrow \infty} (L^2) \int_x^{\infty} e^{i\lambda x} f(x) dx$ , l'intégrale étant convergente dans  $L^2$ . L'auteur étudie, pour  $f(x) \in L^2$ , l'intégrale double

$$(I) \quad J = \int \int e^{i(\alpha x + \beta y)} \frac{f(x) - f(y)}{x - y} dx dy.$$

Pour donner un sens à  $J$ , on la définit par deux intégrations simples successives: fixons une fois pour toutes  $\beta$ , alors

$$(II) \quad K(x, \beta) = \int_{-\infty}^{\infty} e^{i\beta y} \frac{f(x) - f(y)}{x - y} dy$$

est convergente v.p. (v.p. à l'infini et v.p. pour  $y = x$ ) pour presque tous les  $x$  [résultat connu: voir Titchmarsh, Introduction to the Theory of Fourier Integrals, chap. 5, p. 122], et  $K(x, \beta) \in L^2$ . Ensuite on définit  $J$  par

$$(III) \quad \lim_{x \rightarrow \infty} (L^2) \int_x^{\infty} e^{i\alpha x} K(x, \beta) dx.$$

On trouve alors (IV)  $J = -i\pi(\operatorname{sgn} \alpha + \operatorname{sgn} \beta)F(\alpha + \beta)$ . Si de plus  $f(x)$  satisfait à une condition de Lipschitz d'ordre  $\delta > 0$  sur tout intervalle fini, et est sommable, alors  $F(\lambda)$  est partout définie et continue, et (III) devient convergente (C, 1) vers (IV) quels que soient  $\alpha$  et  $\beta$ . Si  $f(x) = Ax/(1+x^2) + g(x)$ ,  $A \neq 0$ ,  $g$  satisfaisant aux conditions ci-dessus, il en est encore ainsi sauf pour  $\alpha = \beta = 0$ . Si enfin  $f'(x)$  existe et est continue et si pour  $|x| \rightarrow \infty$ ,  $f(x) = Ax^{-1} + O(x^{-2})$ ,  $f'(x) = O(x^{-3})$ , alors (III) converge v.p. vers (IV) sauf si  $A \neq 0$ ,  $\alpha = \beta = 0$ . L'a-

uteur signale enfin quelques formules, notamment:

$$\int \int \frac{|f(x) - f(y)|^2}{|x - y|} dx dy = 2\pi \int |x| |F(x)|^2 dx.$$

L. Schwartz (Nancy).

MacDonald, D. K. C. Network analysis involving realizable filter functions. Philos. Mag. (7) 38, 115-131 (1947).

The "response of a filter function  $Y(\omega)$  to a stimulus  $e(t)$ " is the Fourier transform  $i(t)$  of the product of  $Y(\omega)$  and the Fourier transform of  $e(t)$ . The condition of physical realizability is that  $i(t) = 0$  for  $t < 0$ . The author examines  $i(t)$  and related functions of physical interest when  $Y(\omega)$  is a combination of the forms  $(\omega\omega)^{-1} \sin^2 \omega\omega \cdot e^{-i\omega t}$  ( $j^2 = -1$ ), or of derivatives of such functions, with emphasis on the question of physical realizability. His work could be abbreviated by using the fact that  $i(t)$  is the convolution of  $e(t)$  and the Fourier transform of  $Y(\omega)$ .

R. P. Boas, Jr.

Bhatia, A. B., and Krishnan, K. S. Light-scattering in homogeneous media regarded as reflexion from appropriate thermal elastic waves. Proc. Roy. Soc. London. Ser. A. 192, 181-194 (1948).

The main part of the paper is of interest only to physicists. The authors incidentally show that

$$\alpha \sum_{n=-\infty}^{\infty} (n\alpha + \theta)^2 \sin^2 (n\alpha + \theta) = \pi,$$

a result which is of interest because it has apparently been customary in physical literature to "approximate" the series by considering  $\alpha$  to be small and replacing the series by the integral  $\int_{-\infty}^{\infty} x^2 \sin^2 x dx = \pi$ . Two proofs are given, the second of which, based on Poisson's summation formula, is attributed to N. Wiener. [Cf. the following review.]

R. P. Boas, Jr. (Providence, R. I.).

Krishnan, K. S. A simple result in quadrature. Nature 162, 215 (1948).

The author generalizes the result of the paper reviewed above, observing that (formally)

$$\alpha \sum_{n=-\infty}^{\infty} f(n\alpha + \theta) = \int_{-\infty}^{\infty} f(x) dx$$

whenever the Fourier transform of the even function  $f(x)$  vanishes outside  $(-v_0, v_0)$  and  $\alpha \leq 2\pi/v_0$ . The proof is by the Poisson summation formula. Several special cases are mentioned.

R. P. Boas, Jr. (Providence, R. I.).

Crum, M. On a weakly convergent series. J. London Math. Soc. 23, 16-18 (1948).

Let  $f(x)$  be real,  $f''(x) > 0$  ( $a \leq x \leq b$ ), and put  $u = f'(x)$ ,  $F(u) = ux - f(x)$ . Then  $F'(u) = x$ ,  $F''(u) = 1/f''(x)$ , and a formal application of Poisson's formula yields the identity

$$\sum_{n=-\infty}^{\infty} \exp [2\pi i \{ f(x) + (n-x)f'(x) + \frac{1}{2}(n-x)^2 f''(x) \}] = e^{\pi i/4 \{ F''(u) \}} \sum_{n=-\infty}^{\infty} \exp [-2\pi i \{ F(u) + (n-u)F'(u) + \frac{1}{2}(n-u)^2 F''(u) \}].$$

The series diverge, but the author proves that the series, obtained on multiplying by any  $\phi(x)$  of bounded variation in  $(a, b)$  and integrating term by term over  $(a, b)$ , converge and are equal. It is assumed that  $f'''(x)$  is of bounded variation and  $|f'''(x)| > m > 0$ .

G. E. H. Reuter.

**Schwartz, Laurent.** Théorie des distributions et transformation de Fourier. Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.) 23, 7-24 (1948).

A description is given of work in which the distributions introduced by the author [same Ann. Sect. Sci. Math. Phys. (N.S.) 21 (1945), 57-74 (1946); these Rev. 8, 264], especially a special class  $S$  of them called "spherical," are treated in relation to Fourier transforms. A spherical distribution is a continuous linear functional on the suitably topologized space  $S'$  of functions, of  $n$  real variables, which are of class  $C^\infty$  and which together with their derivatives tend to zero at infinity faster than any inverse power of the distance. The Fourier transform maps  $S$  onto  $S$  and  $S'$  onto  $S'$ , and its usual properties remain valid for these domains.

Among various applications, a short proof of the Poisson summation formula is based on the Parseval formula for  $S$  and  $S'$ ; and it is stated that a theorem of Paley and Wiener [Fourier Transforms in the Complex Domain, Amer. Math. Soc. Colloquium Publ., v. 19, New York, 1934, p. 13] is generalized by the result that a spherical distribution is the Fourier transform of a distribution vanishing outside a compact set if it is defined by an analytic function which is integrable near infinity when divided by some power of the distance. A generalization of the Bochner representation theorem is valid: a distribution is "positive definite" only if it is spherical and the Fourier transform of a nonnegative (necessarily spherical) measure. Linear differential equations with constant coefficients whose unknown is a spherical distribution are studied, and in the case of Laplace's equation there are no solutions except the usual harmonic polynomials. A somewhat similar generalization of Fourier transform theory is due to Bochner [Vorlesungen über Fouriersche Integrale, Akademische Verlagsgesellschaft, Leipzig, 1932].

I. E. Segal.

**Hirschman, I. I., Jr.** A new representation and inversion theory for the Laplace integral. Duke Math. J. 15, 473-494 (1948).

Der Verf. beweist mit Hilfe der Post-Widder'schen Umkehrungsformel die folgende neue Inversion der  $L$ -Transformation. Es sei  $\theta(t) = \int_0^t \Phi(u) du$ , und  $\theta(t)$  erfülle die Bedingungen (1)  $\theta(t) = O(t^m)$  wenn  $t \rightarrow 0$  für jeden Wert  $m$ ; (2)  $\theta(t) = O(t^v)$  wenn  $t \rightarrow \infty$  für gewisse Werte  $v$ ; (3)  $\int_0^\infty [\Phi(v) - \Phi(y)] dv = o(|u-y|, u \rightarrow y)$ ; (4)  $f(s) = \int_0^\infty e^{-st} \Phi(t) dt$ ;

$$(5) \quad W_{k,y}^s[f(s)] = \frac{1}{\Gamma(k)} \int_0^\infty (syk)^{1/2} J_k[2(syk)^{1/2}] e^{-sy} f(s) ds;$$

$J_k$  ist die Besselfunktion von der Ordnung  $k$ . Unter diesen Voraussetzungen gilt  $\lim_{k \rightarrow \infty} W_{k,y}^s[f(s)] = \Phi(y)$ . Der Verf. gibt Verallgemeinerungen dieses Satzes und überträgt die Formel auch auf Laplace-Stieltjes'sche Integrale. Er beweist neue Bedingungen für solche Funktionen  $f(s)$ , die durch  $L$ - oder  $L$ -S-Integrale darstellbar sein sollen. Schliesslich gibt er einen neuen Beweis für den Satz von S. Bernstein und D. V. Widder betreffend die Darstellbarkeit einer monotonen Funktion durch ein  $L$ -S-Integral.

W. Saxon (Zürich).

**Hirschman, I. I., Jr., and Widder, D. V.** An inversion and representation theory for convolution transforms with totally positive kernels. Proc. Nat. Acad. Sci. U. S. A. 34, 152-156 (1948).

The authors announce extensions and refinements of results previously obtained by Widder [same Proc. 33, 295-297 (1947); these Rev. 9, 90]. The inversion theorems now

include as special cases the iterated Laplace and Stieltjes transforms. A representation theory has also been constructed by the authors. H. Pollard (Ithaca, N. Y.).

**Feldheim, Ervin.** La transformation de Gauss à plusieurs variables. Application aux polynomes d'Hermite et à la généralisation de la formule de Mehler. Pont. Acad. Sci. Comment. 6, 1-25 (1942).

Let  $\varphi(U)$  be a positive definite quadratic form in the variables  $u_1, \dots, u_n$ ; let  $A$  be the coefficient matrix of  $\varphi$  and  $|A|$  the determinant of  $A$ ; let  $\lambda_1, \dots, \lambda_n$  be parameters and  $\Lambda = \lambda_1 \cdot \lambda_2 \cdot \dots \cdot \lambda_n$ ; finally, let  $R_n$  be the  $n$ -dimensional Euclidean space with coordinates  $u_1, \dots, u_n$  and  $F(U)$  a function defined in  $R_n$ . The author discusses the "Gauss transformation" of  $F(U)$  into

$$\Lambda^{-1} \pi^{-\frac{1}{2}n} |A|^{\frac{1}{2}} \int_{R_n} e^{-\varphi((U-X)/\Lambda)} F(U) dU$$

which is a function of  $x_1, \dots, x_n$ , and also the Gauss transformation based on the reciprocal quadratic form (whose coefficient matrix is  $A^{-1}$ ). He indicates briefly those properties of the  $n$ -dimensional Gauss transformation which are analogous to the known properties of the one-dimensional transformation investigated by F. Tricomi [Ann. Inst. H. Poincaré 8, 111-149 (1938)] and others.

The second half of the paper contains the application of the two Gauss transformations to the theory of Hermite polynomials derived from the quadratic form  $\varphi$ . In particular, the author gives a new derivation of the bilinear generating functions of these polynomials. A. Erdélyi.

**Parodi, Maurice.** Sur la détermination d'une famille de noyaux réciproques. C. R. Acad. Sci. Paris 226, 1877-1878 (1948).

A class of reciprocal kernels  $K(x, t)$  for which the relations

$$g(t) = \int_0^\infty K(x, t) f(x) dx, \quad f(t) = \pm \int_0^\infty K(x, t) g(x) dx$$

are compatible is formally found to have Laplace transforms of the form

$$(1) \quad (\pm 1)^{\frac{1}{2}} \frac{\nu(p)}{\nu[\psi(p)]} e^{-\frac{1}{2}\psi(p)},$$

where  $\nu$  is an arbitrary function and  $\psi(p)$  is periodic of order 2, that is,  $\psi(\psi(p)) = 1$  identically in  $p$ . [Reviewer's remark. Quite apart from rigorous conditions of validity, there ought to be some stipulation to the effect that (1) is in fact a Laplace transform for all nonnegative  $x$ .] The author also mentions a class of similarly constructed kernels in which  $\psi(p)$  is periodic of order  $n$ . A. Erdélyi.

**Aprile, Giuseppe.** Su alcune formule di valutazione nel calcolo operatorio funzionale. Pont. Acad. Sci. Acta 6, 371-374 (1942).

From the transposition theorem

$$f(\Delta + a) V(t) = e^{-at} f(\Delta) \cdot e^{at} V(t), \quad \Delta = d/dt,$$

the author derives the result that if  $f(\Delta) F_u(t) = G(t)$  then

$$\frac{f(\Delta)}{\Delta + a} F_u(t) = e^{-at} \int_0^t e^{as} G(s) ds,$$

and illustrates this by a simple example. A. Erdélyi.

**Stone, W. M.** A list of generalized Laplace transforms. *Iowa State Coll. J. Sci.* 22, 215-225 (1948).

The author derives a list of generalized Laplace transforms considered by Milne-Thomson, Jordan and Samuelson [W. M. Stone, *J. Appl. Phys.* 18, 414-416 (1947); these Rev. 8, 517].

*A. E. Heins* (Pittsburgh, Pa.).

### Polynomials, Polynomial Approximations

**Gagiev, B. M.** Landau's theorem for polynomials. *Uspehi Matem. Nauk* (N.S.) 3, no. 2(24), 229-233 (1948). (Russian)

The author proves by an algebraic method an analogue of Landau's theorem for polynomials. If  $f(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_m z^m + a_{m+1} z^{m+1} + \dots + a_n z^n$ ,  $a_m \neq 0$ , omits the values 0 and 1 in the circle  $|z| < R$ , then

$$R \leq \min \left[ \left\{ \frac{a_0}{a_m} \right\} |a_0/a_m|^{1/m}, \left\{ \frac{a_0}{a_m} \right\} (a_0 - 1)/a_m \right]^{1/m}.$$

The author shows that the inequality is exact if  $a_0$  satisfies a certain condition. His argument that otherwise the inequality is not sharp is open to question. *W. Seidel*.

**Marden, Morris.** A note on lacunary polynomials. *Bull. Amer. Math. Soc.* 54, 546-549 (1948).

The author derives circular bounds for the  $p$  smallest (in modulus) zeros of the polynomials of the lacunary type  $f(z) = a_0 + a_1 z + \dots + a_p z^p + a_{n_1} z^{n_1} + a_{n_2} z^{n_2} + \dots + a_{n_k} z^{n_k}$ ,  $a_0, a_p \neq 0$ ,  $0 < p = n_0 < n_1 < \dots < n_k$ . *E. Frank*.

**Neimark, Yu. L.** The structure of the  $D$ -decomposition of the space of quasipolynomials and the diagrams of Vyšnegradskii and Nyquist. *Doklady Akad. Nauk SSSR* (N.S.) 60, 1503-1506 (1948). (Russian)

The author studies the structure of the decomposition of the space of the parameters  $a_{kz}$ ,  $\tau_j$  of the quasipolynomial  $\sum_{k=0}^m a_{kz} z^k e^{\tau_j z}$  into sets of quasipolynomials with different numbers of roots in the right half of the  $z$ -plane. The parameters  $a_{kz}$  are complex,  $\tau_j$  are real, and  $\tau_m > \tau_{m-1} > \dots > \tau_1 > \tau_0 = 0$ . In addition, the author generalizes results obtained previously [same Doklady (N.S.) 58, 357-360 (1947); 59, 853-856 (1948); these Rev. 9, 348, 428]. The following theorem is proved. To satisfy the conditions, (a) that the boundary of the  $D$ -decomposition of a region  $G$  consist of points of the curve corresponding to the quasipolynomials which allow at least one pure imaginary root; (b) that any two arbitrary points of  $G$  can be joined by a path lying entirely in  $G$  and which intersects the curve of (a) in points whose sum of multiplicities is finite, it is sufficient that the inequality  $|a_{nm}| - |a_{n,m-1}| - \dots - |a_{n0}| > 0$  be satisfied for the points of  $G$ . This inequality is at the same time a necessary and sufficient condition for "roughness" and for the quasipolynomial to belong to the set  $D$  ( $S < \infty$ ). Roughness is defined as follows. Let  $f(z; a_1 \dots a_n)$  be an entire function of  $z$ , continuously dependent on the parameters  $a_j$ . The function is said to be parametrically rough, with respect to the parameters  $a_j$ , at  $a_j = a_j^0$  if for sufficiently small changes in the parameters the number of roots with positive real parts remains unchanged. *S. D. Zeldin*.

**Nikolaev, V. F.** Concerning the approximation of continuous functions by polynomials. *Doklady Akad. Nauk SSSR* (N.S.) 61, 201-204 (1948). (Russian)

Let  $\omega_0(x)$ ,  $\omega_1(x)$ ,  $\omega_2(x)$ ,  $\dots$  be any system of orthogonal polynomials, in an interval  $[a, b]$  and with respect to any

weight  $dg(x)$ . There is then a continuous function whose Fourier development with respect to this system does not converge uniformly. [It is not stated explicitly, but the proof seems to imply that  $\omega_n$  is of degree  $n$ .] A similar result holds for orthogonal trigonometric polynomials in  $[0, 2\pi]$ . Finally, a corresponding result is stated without proof for interpolating polynomials whose fundamental points are the zeros of the polynomials  $\omega_n(x)$ .

*A. Zygmund* (Chicago, Ill.).

**Geronimus, J.** On Gauss' and Tchebycheff's quadrature formulae. *C. R. (Doklady) Acad. Sci. URSS* (N.S.) 51, 655-658 (1946).

Let  $c_0, c_1, \dots, c_n$  be a sequence of real or complex numbers subject to the conditions that

$$\Delta_m = \det \|c_{i+k}\|_{i, k=0, \dots, m} \neq 0, \quad m = 0, 1, \dots, n.$$

Let (1)  $P_m(x) = x^m + \dots$  ( $m = 0, 1, \dots, n$ ) be the set of polynomials enjoying the orthogonality properties

$$S\{P_m(x)P_k(x)\} = \begin{cases} 0, & m \neq k, \\ \Delta_m / \Delta_{m-1}, & m = k; m, k = 0, 1, \dots, n, \end{cases}$$

where the operation  $S$  is defined for polynomials of degree  $2n$  by  $S(\sum a_i x^i) = \sum a_i c_i$ . The author establishes various sets of conditions which insure that  $\{P_m(x)\}$  is identical with the Tchebycheff system

$$(2) \quad P_m(x) = 2^{-m} (x - \alpha + [(x - \alpha)^2 - 4\lambda]^{1/2})^m + 2^{-m} (x - \alpha - [(x - \alpha)^2 - 4\lambda]^{1/2})^m, \quad m = 1, \dots, n; P_0 = 1.$$

The following result is a sample. Let  $x_1^{(m)}, \dots, x_m^{(m)}$  denote the zeros of  $P_m(x)$ , defined by (1), and let

$$(3) \quad \mu_1^{(m)} = m^{-1} \sum_{r=1}^m x_r^{(m)}, \quad (\mu_1^{(m)})^2 = m^{-1} \sum_{r=1}^m (x_r^{(m)})^2.$$

These quantities in general depend on  $m$ ; for the system (2) they are independent of  $m$ . Conversely, if the two means (3) are independent of  $m$ ,  $\mu_1^{(m)} = \mu_1$ ,  $(\mu_1^{(m)})^2 = \mu_1^2$ , say, then the system (1) is identical with the system (2), the connection being via the relations  $\alpha = \mu_1$ ,  $2\lambda = \mu_2^2 - \mu_1^2$ .

*I. J. Schoenberg* (Philadelphia, Pa.).

**Feldheim, Ervin.** Sul prodotto dei polinomi di Laguerre. *Pont. Acad. Sci. Acta* 6, 359-370 (1942).

Expansion of  $L_n^{(\alpha)}(x) L_n^{(\beta)}(x)$  in series of

$$(i) \quad L_{2m}^{(\alpha-\beta)}(x), \quad (ii) \quad \{L_m^{(\alpha)}(x)\}^2, \quad (iii) \quad L_m^{(\gamma)}(x) L_m^{(\delta)}(x),$$

where  $m = 0, 1, \dots, n$ ,  $2\epsilon = \alpha + \beta = \gamma + \delta$ : inversions of these expansions, and other similar expansions, some of them involving Hermite polynomials. *A. Erdélyi* (Edinburgh).

**Reutter, Fritz.** Ueber ganze rationale Funktionen einer dualkomplexen Veraenderlichen. *Bull. École Polytech. Jassy* [Bul. Politehn. Gh. Asachi, Iași] 2, 273-289 (1947).

Polynomials  $f(z) = a_0 + a_1 z + \dots + a_n z^n$  are studied with  $z = x + ey$  and  $a_k = a_k' + ea_k''$ , where  $x, y, a_k'$  and  $a_k''$  are ordinary real numbers, but where  $e^2 = 1$ . Since the corresponding pseudo-Euclidean plane may be considered as derived from the Euclidean plane by means of the imaginary collineation  $x_1' = x_1, x_2' = ix_1, x_3' = x_2$ , the points  $(1, \pm 1, 0)$  correspond to the circular points at infinity, the lines  $y = \pm x$  to the isotropic lines and equilateral hyperbolae to circles. Writing  $f(z) = u(x, y) + ev(x, y)$ , one may define as the zeros of  $f(z)$  the  $n^2$  real or imaginary intersection points  $z_k = x_k + ey_k$

of the curves  $u(x, y) = 0$  and  $v(x, y) = 0$ . Thus

$$[f(z)]^n = \prod_{k=1}^{n^2} (z - z_k).$$

If any  $n$  points  $z_k$  are given, the remaining  $n(n-1)$  points  $z_k$  are determined. If an  $n$ -gon is formed by joining in pairs the  $n$  points  $z_k$  corresponding to any factorization of  $f(z)$ , its  $n(n-1)/2$  sides will be tangent at their midpoints to a curve of class  $n-1$ ; the  $(n-1)^2$  pseudo-Euclidean foci of this curve are the  $(n-1)^2$  zeros of the derivative of  $f(z)$ . The latter result is an exact analogue of Siebeck's theorem for polynomials of an ordinary complex variable. The covariants of cubic and biquadratic polynomials under non-singular linear fractional transformations are determined. The article concludes with a statement of the corresponding results for the dual complex system with  $\epsilon^2 = 0$ .

M. Marden (Milwaukee, Wis.).

### Special Functions

\*Magnus, Wilhelm, und Oberhettinger, Fritz. Formeln und Sätze für die speziellen Funktionen der mathematischen Physik. 2d ed. Springer-Verlag, Berlin, 1948. viii+230 pp.

[For the first edition (1943) cf. these Rev. 9, 183.] Die neue Auflage unterscheidet sich von der ersten durch die Verbesserung einiger Irrtümer und Druckfehler und durch kleinere und grössere Zusätze. Das Kapitel über elliptische Funktionen wurde völlig neu geschrieben; sein Umfang hat sich verdoppelt. Grössere Zusätze finden sich bei der allgemeinen und der konfluenteren hypergeometrischen Funktion, den Zylinderfunktionen und der Gammafunktion. Als Anhang wurde ein Abschnitt über elementare Funktionen angefügt, der einige Fourierreihen und solche Formeln enthält, die bei der Untersuchung der im Hauptteil des Buches behandelten Funktionen von Bedeutung sind.

From the preface.

Williams, John. The summation of a particular type of power series. *Philos. Mag.* (7) 39, 450-454 (1948).

The series  $S(z) = \sum a_n z^n / n!$  with  $\sum_{i=0}^k \lambda_i a_{i+k} = 0$  for all  $n$ , where  $\lambda_0, \dots, \lambda_k$  are  $k$  complex constants with  $\lambda_0 \neq 0, \lambda_k \neq 0$ , is uniformly convergent in the finite  $z$ -plane. The author extends a method of L. Silberstein [same *Mag.* (7) 25, 1003-1004 (1938)] to obtain by the use of finite difference methods a closed expression for  $S(z)$  which involves the zeros of the polynomial  $\sum_{i=0}^k \lambda_i p^i$ . The article concludes with some applications of the result to special cases.

M. J. Gottlieb (Newark, N. J.).

Grammel, R. Eine Verallgemeinerung der Kreis- und Hyperbelfunktionen. *Ing.-Arch.* 16, 188-200 (1948).

The generalised trigonometric functions,  $x = \cos(n)v$  and  $y = \sin(n)v$ , originate from a parametric representation of the curve  $x^n + y^n = 1$  ( $n$  is a positive integer; the parameter is the double area enclosed by the positive  $x$ -axis, the curve and the radius vector). These functions can be defined as inversions of the integrals

$$v = \int_x^1 (1-x^n)^{(1-n)/n} dx = \int_0^y (1-y^n)^{(1-n)/n} dy.$$

From the relations  $dx/dv = -y^{n-1}$ ,  $dy/dv = x^{n-1}$  a nonlinear

differential equation of the second order follows which is satisfied by both functions. The number  $\pi_n = 2 \int_0^1 (1-t^n)^{(1-n)/n} dt$  corresponds to  $\pi$  and is always between  $\pi$  and 4 when  $n > 2$ . An extensive list of formulae for these functions shows much similarity with the corresponding list for ordinary trigonometric functions ( $n = 2$ ). Generalised hyperbolic functions similarly originate from a parametrisation of the curve  $x^n - y^n = 1$ . An extensive list of formulae for these functions is supplemented by formulae exhibiting the connection between the two classes of functions considered here. The case of a nonintegral  $n$  is briefly discussed, and mention is made of further generalisations, obtained by the parametrisation of other curves. A. Erdélyi (Edinburgh).

Grammel, Richard. Eine Verallgemeinerung der Kreis- und Hyperbelfunktionen. *Arch. Math.* 1, 47-51 (1948). A fuller account appeared in the paper reviewed above.

Zagar, F. Sul potenziale di ellissoidi. *Pont. Acad. Sci. Comment.* 10, 407-429 (1946).

The author investigates the functions

$$F_{nn}(k) = \int_{-1/\eta}^{1/\eta} (1-k^2 \cos^2 \varphi)^{n-1} (\cos \varphi)^{2n-2k-1} d\varphi$$

and

$$F_{nn}(\epsilon, \eta) = \int_0^{2\pi} (\cos \lambda)^{2n-2k-2} F_{nn} \left( \left| \epsilon^2 - \eta^2 \frac{1-\lambda^2}{1-\eta^2} \sin^2 \lambda \right|^{\frac{1}{2}} \right) d\lambda.$$

He obtains recurrence relations and other auxiliary formulae, gives explicit expressions for some of the functions, and explains their application to the computation of the potential of an homogeneous ellipsoid. The potential of certain nonhomogeneous ellipsoids and the mutual attraction of two ellipsoids are mentioned as further problems which can be solved by similar methods. A. Erdélyi.

Pinney, Edmund. On a note of Galbraith and Green. *Bull. Amer. Math. Soc.* 54, 527 (1948).

The author observes that the integral

$$\frac{1}{\pi} \int_0^\pi \left\{ \frac{1-r^2}{1-2r \cos \theta + r^2} \right\}^{n+1} d\theta,$$

evaluated, for  $n > -1$ , by A. S. Galbraith and J. W. Green [same *Bull.* 53, 314-320 (1947); these Rev. 8, 511], may be identified, for all  $n$ , with Laplace's second integral for the Legendre function, and gives asymptotic formulae valid for  $\Re(n) \neq -\frac{1}{2}$ . L. S. Bosanquet (London).

Toscano, Letterio. Trasformata di Laplace di prodotti di funzioni di Bessel e polinomi di Laguerre. Relazione integrale su funzioni ipergeometriche più generali della  $F_A$  di Lauricella. *Pont. Acad. Sci. Comment.* 5, 471-500 (1941).

The author computes the "Laplace transforms" of

$$t^a \prod_{m=1}^n J_{am}(c_m t), \quad t^a \prod_{m=1}^n J_{am}(c_m t^2), \quad t^a \prod_{m=1}^n L_{rm}^{(am)}(c_m t^2)$$

and of other similar combinations. He considers various special cases of interest. He introduces and studies the function

$$F(\alpha; \beta_i; \gamma_i; x_i) = \sum_{k_1, \dots, k_n=0}^{\infty} \frac{\Gamma(\alpha + mk_1 + \dots + mk_n)}{\Gamma(\alpha)} \times \prod_{i=1}^n \frac{\Gamma(\gamma_i) \Gamma(\beta_i + k_i)}{\Gamma(\gamma_i + k_i) \Gamma(\beta_i)} \frac{x_i^{mk_i}}{k_i!}$$

which is a generalisation of Lauricella's  $F_A$ , and notes the relation

$$\int_0^1 x^{\gamma-1} (1-x)^{\alpha-\gamma-1} F(\alpha; \beta_i; \gamma_i; xx_i) dx \\ = \frac{\Gamma(\gamma) \Gamma(\alpha-\gamma)}{\Gamma(\alpha)} F(\gamma; \beta_i; \gamma_i; x_i).$$

The convergence of  $F$  is not investigated. *A. Erdélyi.*

**Bordoni, Piero Giorgio.** *Sulle funzioni di Stokes.* Pont. Acad. Sci. Comment. 9, 87-113 (1945).

The functions investigated here are connected with spherical Bessel functions. The "Stokes functions" of the first kind are

$$f_m(ix) = i^{-m-1} e^{i\alpha} (\frac{1}{2}\pi x)^{\frac{1}{2}} H_{m+1}^{(2)}(x) = {}_2F_0(-m, m+1; (2ix)^{-1}),$$

the functions of the second kind

$$F_m(ix) = (2m+1)^{-1} ix \{ m f_{m-1}(ix) + (m+1) f_{m+1}(ix) \},$$

and those of the third kind  $\xi_m(ix) = f_m(ix)/F_m(ix)$ . The author derives various formulae for these functions, gives explicit expressions of the functions of the first two kinds for  $m=0, 1, 2, 3, 4, 5$ , and of those of the third kind for  $m=0, 1, 2, 3$ ; and, for the same functions, he also gives graphs and short numerical tables, the latter for  $0.10 \leq x \leq 30.0$  at various intervals. *A. Erdélyi.*

**MacRobert, T. M.** *On an identity involving  $E$ -functions.* Philos. Mag. (7) 39, 466-471 (1948).

For his  $E$ -function (which is, essentially, a generalised hypergeometric function), the author proves the identity

$$E(p; \alpha_r; q; \rho_s; z) = \sum_{k=0}^n A_k s^{-k} E(p; \alpha'_r; q; \rho'_s; z),$$

in which the  $A_k$  are explicitly determined constant coefficients (gamma products),  $\alpha'_r = \alpha_r - n + k$ ,  $\alpha'_s = \alpha_s + n$ ,  $\alpha'_r = \alpha_r + k$  ( $r = 3, 4, \dots, p$ ); and  $\beta'_s = \beta_s + k$ . By specialising the parameter values, multiple integrals involving Bessel functions can be evaluated in terms of the  $E$ -function, and various expansions involving Bessel and hypergeometric functions can be deduced. *A. Erdélyi* (Edinburgh).

### Harmonic Functions, Potential Theory

**Heins, Maurice.** *On some theorems associated with the Phragmén-Lindelöf principle.* Ann. Acad. Sci. Fenniae. Ser. A. I. Math.-Phys. no. 46, 10 pp. (1948).

L'auteur étend les résultats d'Ahlors [Trans. Amer. Math. Soc. 41, 1-8 (1937)] aux fonctions  $u(z) \neq -\infty$  sous-harmoniques dans le rectangle  $[a < x < b, |y| < \frac{1}{2}\pi]$ , et satisfaisant à  $\limsup u(x+iy) \leq 0$  lorsque  $y \rightarrow \pm \frac{1}{2}\pi$ . Ces résultats sont basés sur la convexité de la norme de Nevanlinna  $m(x, u) = \int_{-\pi/2}^{\pi/2} u(x+iy) \cos y dy$  par rapport à la famille de fonctions  $\alpha e^x + \beta e^{-x}$  ( $A$ -convexité). Réciproquement toute fonction  $\varphi(x)$   $A$ -convexe peut être considérée comme la norme d'une fonction de la famille. Même propriété de convexité pour la norme de Carleman  $[\int_{-\pi/2}^{\pi/2} \{u(x+iy)\}^2 dy]^{\frac{1}{2}}$  en supposant  $u$  non négative. Transposition et application aux fonctions sous-harmoniques dans le demi-plan. Propriété de croissance de la norme de Nevanlinna. *J. Lelong.*

**Lelong, Jacqueline.** *Propriétés des fonctions surharmoniques positives dans un demi-espace.* C. R. Acad. Sci. Paris 226, 1161-1163 (1948).

**Lelong, Jacqueline.** *Quelques applications de la théorie du potentiel.* C. R. Acad. Sci. Paris 226, 1333-1335 (1948).

**Lelong, Jacqueline.** *Distributions capacitaires pour les potentiels de fonction de Green.* C. R. Acad. Sci. Paris 226, 1500-1502 (1948).

Let  $D$  be the half-space  $x > 0$  in Euclidean  $p$ -space  $R^p$ ,  $p \geq 2$ , bounded by the hyperplane  $\pi$ , and let  $u(M)$  be a positive superharmonic function in  $D$ . The first note announces, with indications of proofs, theorems on the behavior of  $u(M)/x$  as  $M$  approaches a point of  $\pi$ . The full statements involve several definitions and are too long to reproduce here; one of the principal theorems is that  $\liminf_{M \rightarrow \infty} u(M)/x = c$  is necessarily finite, and that  $u(M)/x \rightarrow c$  as  $M \rightarrow \infty$  radially, except at most for a set of rays which cut a set of zero capacity from the unit sphere. The second note gives the applications of these results to theorems of the Phragmén-Lindelöf type and also gives further results on the limiting behavior of "potentials of Green," i.e., the potentials of positive mass distributions with respect to the Green's function of  $D$ . The third note applies these results to the problem of characterizing the subsets of  $D$  (allowed to have limit points on the boundary of  $D$ ) which have capacity distributions [in the terminology of H. Cartan, Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.) 22, 221-280 (1946); these Rev. 8, 581] for the Green's function of  $D$ . [The reviewer would like to remark that the result concerning the limit of  $u(M)/x$  is also contained in an independent forthcoming paper by Ahlfors and M. Heins.]

*R. P. Boas, Jr.* (Providence, R. I.).

**Reuter, G. E. H.** *An inequality for integrals of subharmonic functions over convex surfaces.* J. London Math. Soc. 23, 56-58 (1948).

L'auteur étend et simplifie une note de R. M. Gabriel [même J. 21, 87-90 (1946); ces Rev. 8, 461]. Si  $u$  est une fonction non négative et sous-harmonique, continue dans le domaine limité par une surface convexe  $C$  et sur cette surface, on a, si  $\Gamma$  est une sphère intérieure au domaine,  $\int_{\Gamma} u(P) dS \leq 2 \int_C u(P) dS$ . La démonstration est valable dans un espace à un nombre quelconque de dimensions; elle repose sur une majoration classique de la mesure harmonique d'une portion de surface convexe par rapport à  $P$  par le nombre  $(2/k)\Omega$ ,  $k$  étant la surface de la sphère de rayon 1,  $\Omega$  l'angle solide sous lequel de  $P$  on voit la portion de surface. [Voir, par exemple, Nevanlinna, Eindeutige analytische Funktionen, Springer, Berlin, 1936, p. 68.] Il est indispensable que la surface soit convexe, et le nombre 2 ne peut pas être amélioré;  $u$  n'a pas besoin d'être continue.

*L. Schwartz* (Nancy).

**Potts, D. H.** *A note on the operators of Blaschke and Pivaloff.* Bull. Amer. Math. Soc. 54, 782-787 (1948). For the function  $f(P) = f(x, y)$ , define the mean values

$$L(f; P; r) = (2\pi r)^{-1} \int_{C(P; r)} f(Q) dS_Q,$$

$$A(f; P; r) = (\pi r^2)^{-1} \iint_{D(P; r)} f(Q) dQ,$$

where  $C(P; r)$  and  $D(P; r)$  are the perimeter and interior, respectively, of the circle with center  $P$  and radius  $r$ . The

Blaschke and Privaloff (generalized Laplace) operators are defined to be

$$\nabla_R f(P) = \lim_{r \rightarrow 0} 4r^{-2} [L(f; P; r) - f(P)],$$

$$\nabla_a f(P) = \lim_{r \rightarrow 0} 8r^{-2} [A(f; P; r) - f(P)],$$

respectively. The author extends some results obtained by Blaschke, Privaloff and Saks. A typical result is the following. If  $f(P)$  is continuous on the closed disc  $\bar{D}(Q; r)$ , and if  $\nabla_a f(P)$  exists on the open disc  $D(Q; r)$ , then  $8r^{-2} [A(f; Q; r) - f(Q)]$  lies between the upper and lower bounds of  $\nabla_a f(P)$  on  $D(Q; r)$ .

M. O. Reade.

Topolyanskil, D. B. On an estimate of the Dirichlet integral. *Akad. Nauk SSSR. Prikl. Mat. Meh.* 11, 551–554 (1947). (Russian)

Let  $L$  be a smooth rectifiable Jordan curve bounding a finite plane domain  $D$ , let  $\psi_0$  be a continuous function on  $L$ , and let  $\psi^*$  be the function harmonic in  $D$ , continuous in  $D+L$ , and satisfying  $\psi^* = \psi_0$  on  $L$ . Clearly  $\psi^*$  is a member of the family  $S$  of functions  $\psi$  harmonic in  $D$ , continuous in  $D+L$ , and satisfying

$$\int_L (\psi_0 - \psi) (\partial \psi / \partial n) ds = 0,$$

where  $n$  denotes the exterior normal to  $D$ . It is shown that in the family  $S$  the function  $\psi^*$  maximizes the Dirichlet integral for  $D$ ; that is, if  $\psi$  is a member of  $S$  then

$$\int \int_D (\partial^2 \psi / \partial x^2 + \partial^2 \psi / \partial y^2) dA \leq \int \int_D (\partial^2 \psi^* / \partial x^2 + \partial^2 \psi^* / \partial y^2) dA.$$

E. F. Beckenbach (Los Angeles, Calif.).

Levi, G. [Lewy, H.]. On the non-vanishing of the Jacobian in certain one-to-one mappings. *Uspehi Matem. Nauk* (N.S.) 3, no. 2(24), 216–219 (1948). (Russian)

Translated from *Bull. Amer. Math. Soc.* 42, 689–692 (1936).

### Differential Equations

Lyon, W. V. Method of solving differential equations with constant coefficients. *J. Franklin Inst.* 246, 159–164 (1948).

A particular electrical circuit is solved completely by an efficient use of classical (i.e., nonoperational) methods.

P. Franklin (Cambridge, Mass.).

Kartvelišvili, N. A. On conditions for the oscillation of an automatic regulator. *Doklady Akad. Nauk SSSR* (N.S.) 61, 21–23 (1948). (Russian)

The physical problem leads to a linear differential equation with constant coefficients which is solved operationally. To ascertain the character of the solution, oscillatory or not, it is necessary to examine the algebraic nature of the characteristic roots. R. Bellman (Stanford University, Calif.).

Dieulefait, Carlos E. Direct integration of differential equations. *An. Soc. Ci. Argentina* 145, 259–280 (1948). (Spanish)

Continuation of the author's studies [same An. 139, 147–151 (1945); these Rev. 7, 13].

Barber, N. F., and Ursell, F. The response of a resonant system to a gliding tone. *Philos. Mag.* (7) 39, 345–361 (1948).

This paper is concerned principally with the response of a linear resonant system to a force having a frequency (defined in the usual way) which varies linearly with the time. The mathematical theory possesses nothing of special interest, and the value of the paper is to be found in the extensive physical discussion of the results. The authors discuss particularly the bearing of the results on a certain method for the experimental harmonic analysis of waves. The paper closes with a brief consideration of the situation in which a system, with a resonance frequency which varies with time, is subjected to a force having a fixed frequency.

L. A. MacColl (New York, N. Y.).

Sobol', I. M. On the asymptotic behavior of the solutions of linear differential equations. *Doklady Akad. Nauk SSSR* (N.S.) 61, 219–222 (1948). (Russian)

Consider the equation  $y^{(n)} = \sum_{j=1}^n a_j(t) y^{(n-j)}$ , where

$$(1) \quad \int_s^\infty |a_j(t)| t^{j-1} dt < \infty.$$

Set

$$(2) \quad \psi(x) = \sum_{j=1}^n \int_s^x |a_j(t)| t^{j-1} dt.$$

The author proves that if (1) is satisfied, there exists a fundamental set of solutions  $y_s(x)$ ,  $0 \leq s \leq n-1$ , having the form

$$(3) \quad y_s(x) = x^s + O\left(\int_s^x \cdots \int_s^t \psi(t_i) dt_i\right).$$

He also considers the case where the differential equation above has a nonhomogeneous term  $b(t)$ . These results are extensions of results of Wilkins [Bull. Amer. Math. Soc. 50, 388–394 (1944); these Rev. 5, 265] and of Haupt.

R. Bellman (Stanford University, Calif.).

Wazewski, T. Sur la limitation des intégrales des systèmes d'équations différentielles linéaires ordinaires. *Studia Math.* 10, 48–59 (1948).

The paper deals with the differential system

$$y_i'(x) = \sum_{j=1}^n a_{ij}(x) y_j(x) + b_i(x), \quad i = 1, \dots, n,$$

on a (finite or infinite) interval  $(\alpha, \beta)$  upon which the coefficients are continuous. If  $y_i(x)$ ,  $i = 1, \dots, n$ , denotes the integral for which  $y_i(\xi) = \eta_i$ ,  $i = 1, \dots, n$ , with  $\alpha < \xi < \beta$ , the primary result is the following theorem. If  $s(x)$  and  $S(x)$  are respectively the smallest and the largest characteristic roots of the Hermitian form  $\sum_{i,j=1}^n [a_{ij}(x) + \bar{a}_{ji}(x)] \xi_i \bar{\xi}_j$ , and if the functions  $t(x)$ ,  $T(x)$ ,  $l(x)$ ,  $L(x)$  are continuous on  $(\alpha, \beta)$  and such that  $t(x) \leq s(x)$ ,  $S(x) \leq T(x)$ ,  $l(x) \leq -(\sum_i |b_i(x)|^2)^{1/2}$ ,  $L(x) \geq \{\sum_i |b_i(x)|^2\}^{1/2}$ , then  $\Phi(x) \leq [\sum_i |y_i(x)|^2]^{1/2} \leq \varphi(x)$ , for  $\alpha < x \leq \xi$ ;  $\varphi(x) \leq [\sum_i |y_i(x)|^2]^{1/2} \leq \Phi(x)$ , for  $\xi \leq x < \beta$ , where  $\varphi(x)$  and  $\Phi(x)$  are respectively the integrals of the equations  $\varphi' = t(x)\varphi + l(x)$ ,  $\Phi' = T(x)\Phi + L(x)$ , which at  $\xi$  take the value  $\{\sum_i |\eta_i|^2\}^{1/2}$ .

R. E. Langer (Madison, Wis.).

Freilich, Gerald. Note on the eigenvalues of the Sturm-Liouville differential equation. *Bull. Amer. Math. Soc.* 54, 405–408 (1948).

For the case of the Sturm-Liouville system

$$(1) \quad (pu')' + (\lambda p - g)u = 0, \quad a \leq x \leq b, \\ u(a) = c_1 u(b), \quad u'(a) = c_2 u'(b), \quad c_1 c_2 p(b) = p(a),$$

with

$$p(x) \geq m > 0, \quad q(x) \geq 0, \quad \beta \geq \rho(x) \geq \alpha > 0,$$

the paper gives the following characterization of the  $n$ th eigenvalue  $\lambda_n$ . Let  $v_1, \dots, v_n$  be any set of linearly independent functions which satisfy the boundary conditions of (1), and  $u = c_1 v_1 + \dots + c_n v_n$ . Further let

$$g(v_1, \dots, v_n) = \max \left\{ \int_a^b (pu' u' + quu) dx / \int_a^b pu^2 dx \right\}.$$

Then  $\lambda_n = \min g(v_1, \dots, v_n)$ . It is indicated how this characterization may be used to obtain certain comparison relations and to approximate  $\lambda_n$ .

R. E. Langer (Madison, Wis.).

Imai, Isao. On a refinement of the W.K.B. method.

Physical Rev. (2) 74, 113 (1948).

The matter at issue is the differential equation

$$d^2\Phi/dx^2 + k^2 P(x)\Phi = 0,$$

in which  $k$  is a large parameter and  $P(x)$  has a simple zero on the interval considered. It was shown by the reviewer that if  $z = \int P^{\frac{1}{2}} dx$ , and  $Z_1$  is any Bessel function of the order  $\pm \frac{1}{2}$ , the expressions (2)  $P^{-\frac{1}{2}} z^{\frac{1}{2}} Z_1(kz)$  are solutions of a differential equation which resembles (1) in a specific way. From this resemblance it was shown that the solutions of (1) are representable asymptotically as to  $k$  by the functions (2). It is shown in this paper that with  $\rho$  and  $\lambda$  as suitably chosen constants, and with  $\xi = (3z)^{\frac{1}{2}} + \lambda \rho^{-\frac{1}{2}}$ , the expressions  $P^{-\frac{1}{2}} z^{\frac{1}{2}} Z_1(\xi \rho^{\frac{1}{2}})$  are solutions of a differential equation which, in the immediate neighborhood of the zero of  $P(x)$ , resembles (1) more closely than does the equation solved by the functions (2). The question of the extent to which the representation of the solutions of (1) is improved by the use of (3) in place of (2) is not touched upon. It may reasonably be suspected that the improvement will hardly offset the greater complexity of the formulas.

R. E. Langer (Madison, Wis.).

Strutt, M. J. O. On Hill's problems with complex parameters and a real periodic function. Proc. Roy. Soc. Edinburgh. Sect. A. 62, 278-296 (1948).

"Hill's differential equation  $\omega''(z) + [\lambda + \gamma \Phi(z)] \omega(z) = 0$  derives its importance from being the prototype of the different equations of Lamé and of the equation of Mathieu, which are connected with wave and potential problems in mathematical physics. Besides this, numerous instances of its occurrence in problems of elasticity and of dynamical and statical stability are known." Here  $\lambda$  and  $\gamma$  are (real or complex) parameters,  $\Phi(z)$  is a real periodic function (with real period  $\delta$ ) of the real variable  $z$ . If two-point boundary conditions of the type  $\omega(z + \delta) = \omega(z)$ ,  $\omega'(z + \delta) = \omega'(z)$  are prescribed and the ratio of  $\lambda$  and  $\gamma$  is given, the author speaks of Hill's boundary value problem. The case of real parameters has already been investigated: in this paper the author considers complex parameters. The investigations are based largely on the Green's function of Hill's boundary value problem. Bounds for the characteristic values are derived. Asymptotic forms of the characteristic functions are used to derive important information about the location of characteristic values. The expansion of an arbitrary function in terms of characteristic functions is obtained by the function-theoretical method. A. Erdélyi (Edinburgh).

Fichera, Gaetano. Sulle condizioni necessarie e sufficienti per l'integrabilità in grande delle forme differenziali esterne. Matematiche, Catania 2, 20-24 (1946). Fichera, Gaetano. Sull'integrazione in grande delle forme differenziali esterne di qualsivoglia grado. Univ. Nac. Tucumán. Revista A. 6, 51-70 (1947).

In Euclidean  $r$ -space let  $D$  be an open region plus its boundary. Let  $\Pi$  be a given exterior differential form of degree  $k+1$  with coefficients of class  $C'$  in  $D$ . Let  $\Omega$  be an unknown exterior differential form of degree  $k$  which is to be determined so as to satisfy  $\Omega' = \Pi$ , where  $\Omega'$  is the derived form of  $\Omega$ . The author defines regular, closed variety  $V_{k+1}$  and simple connection. He states and gives proofs of the following theorems: (1) there exists an  $\Omega$  if and only if  $\int \Pi = 0$  over every regular closed  $V_{k+1}$  in  $D$ ; (2) if  $D$  has simple  $(k+1)$ -dimensional connection, there exists an  $\Omega$  if and only if  $\Pi' = 0$ . The first paper is a synopsis of the second [cf. also Ricerca Sci. 16, 1117-1119 (1946); Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 540-543 (1946); these Rev. 8, 382].

J. M. Thomas.

Levi, G. [Lewy, H.]. A priori limitations for solutions of Monge-Ampère equations. Uspehi Matem. Nauk (N.S.) 3, no. 2(24), 191-215 (1948). (Russian)

Translated from Trans. Amer. Math. Soc. 37, 417-434 (1935).

Mitrinovich, D. S. Sur une équation linéaire aux dérivées partielles. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. 1, 168-181, 209-226 (1946). (Serbian. French summary)

The paper was summarized in C. R. Acad. Sci. Paris 210, 783-785 (1940); these Rev. 2, 364.

Friedrichs, K. O. Nonlinear hyperbolic differential equations for functions of two independent variables. Amer. J. Math. 70, 555-589 (1948).

The system of differential equations

$$(*) \quad \sum_{i=1}^N \left( a^{mn} \frac{\partial u^i}{\partial x} + b^{mn} \frac{\partial u^i}{\partial y} \right) = g^m, \quad m = 1, \dots, N,$$

where the matrices  $a = \{a^{mn}\}$ ,  $b = \{b^{mn}\}$ , and  $g = \{g^m\}$  depend on  $x$ ,  $y$ , and  $u^1, \dots, u^N$ , is said to be a hyperbolic system if there exist matrices  $p$  and  $q$  with nonvanishing determinants such that  $p b q = \{ \delta_i^j \}$ ,  $p a q = \{ k^i \delta_i^j \}$ . The eigenvalues  $k^1, \dots, k^N$  of the matrix  $a - kb$  are assumed to be real. If the  $a^{mn}$ ,  $b^{mn}$ ,  $g^m$  in the hyperbolic system (\*) possess continuous second derivatives with respect to  $x$ ,  $y$ ,  $u^1, \dots, u^N$ , the author shows that (\*) possesses in a neighborhood  $R$  of a segment  $I$  of the  $x$ -axis a unique solution  $\{u^1(x, y), \dots, u^N(x, y)\}$  which assumes given initial values  $\{\bar{u}^1(x), \dots, \bar{u}^N(x)\}$  on  $I$ , provided the  $\bar{u}^i(x)$  possess continuous second derivatives. The solution is proved to possess continuous second derivatives.

F. G. Dressel (Durham, N. C.).

Galonen, L. M. Sur l'intégration formelle de quelques équations aux dérivées partielles du second ordre. C. R. (Doklady) Acad. Sci. URSS (N.S.) 55, 283-286 (1947).

For the equation  $F(z, p, q, r, s, t) = 0$ , where  $z = z(x, y)$  and  $p, q, r, s, t$  are partial derivatives of the first and second order, the author presents a method of finding particular solutions when  $p$  and  $q$  are thought of as functions of  $z$  alone. Also the paper discusses methods of obtaining, from particular solutions containing arbitrary constants, solutions

containing arbitrary functions. In some cases these methods lead to the general integral of  $F(z, p, q, r, s, t) = 0$ .

F. G. Dressel (Durham, N. C.).

**Janet, Maurice.** *Sur un système simple d'équations du second ordre.* Ann. Soc. Polon. Math. 20 (1947), 335-346 (1948).

The single unknown  $z$  is a function of two independent variables  $x, y$ . Monge's notation for derivatives is employed. The system  $S_1: r = \alpha p, 2s = \beta p + \gamma q, t = \delta q$ , where  $\alpha, \beta, \gamma, \delta$  are known functions of  $x, y$ , is equivalent to a system of total differential equations  $S_1$  in the three unknowns  $z, p, q$ . Consequently, the solution of  $S_1$  involves at most three arbitrary constants (the initial values of  $z, p, q$ ) and this maximum is attained only if  $S_1$  is completely integrable. The condition for complete integrability is found to be a system  $S_2$  of four partial differential equations of the first order and first degree in  $\alpha, \beta, \gamma, \delta$ . Regarding  $S_2$  as a system for determining unknown  $\alpha, \beta, \gamma, \delta$ , the author reduces its integration to that of two equations of the second order: the first is in the unknown  $\beta/\gamma$  and is of the Liouville type; the second is in the unknown  $\beta/\gamma$  and can be integrated explicitly by a process noted in studying the system  $S_1$  which expresses the condition that the general solution of  $S$  depends on exactly two arbitrary constants. Finally,  $S_2$  is applied to form the conditions that the ordinary equation  $y' = f(x, y)$  admits a one-parameter group of the form  $x^* = X(x), y^* = Y(y)$ . J. M. Thomas (Durham, N. C.).

**Bergman, S., and Schiffer, M.** *Kernel functions in the theory of partial differential equations of elliptic type.* Duke Math. J. 15, 535-566 (1948).

The authors investigate the following equation:

$$(*) \quad \Delta\varphi(Z) = P(Z)\varphi(Z), \quad \Delta\varphi = \partial^2\varphi/\partial x^2 + \partial^2\varphi/\partial y^2.$$

A point of a finite domain  $B$  bounded by  $r$  smooth curves  $C_r, r = 1, \dots, n$ , which form the boundary  $C = \sum_{r=1}^n C_r$  of  $B$ , will be denoted by a capital letter, e.g.  $Z, W, V$ . In particular, if  $Z$  has the Cartesian coordinates  $x, y$ , the authors write  $Z = (x, y)$  and  $z = x + iy$  denotes the corresponding complex number. The function  $P(z)$  is continuous in the closed domain  $B + C$  and  $P(Z) > 0$  if  $Z \in B + C$ . Besides the equation  $(*)$  the authors investigate the Dirichlet integral

$$D[\varphi, \varphi] = \iint_B [(\text{grad } \varphi)^2 + P\varphi^2] dx dy.$$

Equation  $(*)$  is the Euler-Lagrange equation for the variation problem  $\delta D[\varphi, \varphi] = 0$ . The authors call  $S(Z, W)$  a fundamental solution of  $(*)$  with respect to  $B$  if it is continuous and continuously differentiable with respect to  $Z$  everywhere in  $B$  except for the point  $W$  and if it satisfies  $(*)$  everywhere except at  $W$ . At  $Z = W$ , however,  $S(Z, W)$  shall become logarithmically infinite. In particular, there are the following two important fundamental solutions: (a) Green's function  $G(Z, W)$  which vanishes if  $Z \in C$ , (b) Neumann's function  $N(Z, W)$  with vanishing normal derivatives on  $C$ . These functions together define the kernel function  $K(Z, W) = N(Z, W) - G(Z, W)$ , which is symmetric in  $Z$  and  $W$ , a solution of  $(*)$  in  $B$  and finite even for  $Z = W$ . The authors obtain the development  $(**)$   $K(Z, W) = \sum_{s=1}^{\infty} \varphi_s(Z) \varphi_s(W)$  converging uniformly in each closed subdomain of  $B$ . The development  $(**)$  leads to applications to the numerical treatment of the theory of  $(*)$  as well as to new theoretical consequences. The authors prove the following new properties for the kernel function: it is nonnegative in  $B$ ; every

solution of  $(*)$  which is nonnegative on  $C$  is nonnegative in  $B$ ; the kernel function is subharmonic in  $B$ . The fundamental solution  $R(Z, W)$  of  $(*)$  in  $B$  which satisfies on  $C$  the boundary condition  $\partial R/\partial n_Z = \lambda(Z)R(Z \in C, W \in B)$  is called Robin's function and plays a role in the boundary problem of the third kind analogous to that of Green's function in the problem of the first kind and Neumann's function in that of the second kind ( $\lambda(Z)$  is a continuous nonnegative function on  $C$ ). Each Robin's function is nonnegative in  $B$ . For two Robin's functions  $R_1(Z, W)$  and  $R_2(Z, W)$  the authors obtain  $R_1(Z, W) \geq R_2(Z, W)$ , if  $\partial R_i/\partial n_Z = \lambda_i R_i(Z \in C, W \in B, i = 1, 2)$ ;  $\lambda_1(Z) \geq \lambda_2(Z)$  on  $C$ .

The variation of the fundamental solutions and of the kernel function is studied when the coefficient of the differential equation varies. The results are applied to the study of the nonlinear partial differential equation  $\Delta\varphi = P(Z; \varphi) \cdot \varphi$  of elliptic type;  $P(Z; t)$  is continuous for  $Z \in B + C$  and satisfies a Lipschitz condition  $|P(Z; t) - P(Z; t_1)| \leq \kappa |t - t_1|$ . The authors suppose furthermore that  $0 < m \leq P(Z; t) \leq M$  ( $Z \in B + C, t$  real) and seek a solution  $\varphi(Z)$  which assumes prescribed continuous boundary values  $\varphi(Z)$  on  $C$ . Every integrand which differs from the integrand of  $D[\varphi, \varphi]$  by a divergence term leads to the same variational equation  $(*)$ . Thus the authors introduce two arbitrary functions  $a(Z)$  and  $b(Z)$  which are continuous and continuously differentiable in  $B + C$ , suppose furthermore that on  $C$   $a(Z) \cos(n, x) + b(Z) \cos(n, y) = -\lambda(Z) < 0$ , and consider the integral

$$\Theta(\varphi, \psi) = \iint_B \left[ \frac{\partial}{\partial x} (a \varphi \psi) + \frac{\partial}{\partial y} (b \varphi \psi) \right] dx dy.$$

They obtain representations for new fundamental solutions. Finally certain of the initial assumptions of the theory are weakened and the effect on the theory is investigated.

M. Pinl (Cologne).

**\*Leutert, Werner.** *Die erste und zweite Randwertaufgabe der linearen Elastizitätstheorie für die Kugelschale.* Thesis, Eidgenössische Technische Hochschule in Zürich, 1948. 44 pp.

The fundamental equation of linearized elasticity theory for a homogeneous isotropic material is

$$(1) \quad \Delta \varrho + (1/(1-2\mu)) \text{grad div } \varrho + \mathbf{v}/G = 0,$$

where  $\varrho$  is the vector of displacement,  $\mathbf{v}$  the vector of force per unit volume, and  $\mu$  and  $G$  are (scalar) constants. In case of given displacements on the surface of the elastic body we have the first boundary value problem, in case of given stresses on the surface the second boundary value problem. In the present thesis the author sets out to solve these two problems of equation (1) for a spherical shell. At first he deals with the homogeneous equation (1a) in which  $\mathbf{v} = 0$ . For this equation, he constructs for every nonnegative integer  $n$  a system of  $6n+3$  linearly independent solutions, and also a second system of  $6n+3$  (in the case  $n=1$  only 6) solutions. The construction follows E. T. Whittaker's construction of solutions of the potential equation [Math. Ann. 57, 333-355 (1902)] and the solutions are in fact linear combinations of products of powers of  $r$ , associated Legendre functions, and trigonometric functions. He shows that his first (second) system of solutions can be used to solve the first (second) boundary value problem of equation (1a) for the inside or outside of a sphere and also for a spherical shell. The convergence of the infinite series

representing the solution is briefly discussed as is also equation (1). The solution for a clamped hemispherical shell is briefly indicated.

A. Erdélyi (Edinburgh).

Pleijel, Åke. On Hilbert-Schmidt's theorem in the theory of partial differential equations. *Kungl. Fysiografiska Sällskapets i Lund Förfhandlingar* [Proc. Roy. Physiog. Soc. Lund] 17, no. 2, 11 pp. (1946).

The author discusses the construction of the Green's function  $G(P, \pi)$  for the eigenvalue problem defined on a region  $S$  (not bounded) of the  $(x, y)$ -plane by the differential equations  $(\Delta - q)u + \lambda ku = 0$  and the boundary conditions:  $u$  or  $\partial u / \partial n$  vanishes on the "boundary" of  $S$  in a certain sense. Under the same assumptions made in his previous article [Ark. Mat. Astr. Fys. 30A, no. 21 (1944); these Rev. 6, 229] the author proves the following form of the Hilbert-Schmidt development theorem by applying the spectral theorem to the integral operator

$$u = Kf = \int \int_S G(P, \pi) k(P) f(P) dP$$

in the Hilbert space  $\mathfrak{D}$ . If  $f$  is an element of  $\mathfrak{D}$ , every function in the range of  $K$  admits a development  $u = \sum u_{\omega}$ , where the  $u_{\omega}$  are regular functions (i.e., the differential operator is defined) belonging to the eigenspaces  $\mathfrak{D}(\omega)$ . The convergence is uniform on every closed subregion of  $S$ . If  $k(x, y) \geq 0$  the theorem holds if  $f$  is an arbitrary function subject only to the conditions  $\iint_S k f^2 dx dy < \infty$ .

M. J. Gottlieb (Newark, N. J.).

Amerio, Luigi. Sull'integrazione delle equazioni lineari a derivate parziali del secondo ordine di tipo ellittico. *Pont. Acad. Sci. Acta* 9, 213-227 (1945).

This is essentially the same paper that appeared in Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 175-182 (1946) [these Rev. 8, 383], and which was later published in fuller detail in Amer. J. Math. 69, 447-489 (1947) [these Rev. 9, 37].

F. G. Dressel.

Tranter, C. J. The use of the Mellin transform in finding the stress distribution in an infinite wedge. Quart. J. Mech. Appl. Math. 1, 125-130 (1948).

Mathematically, the problem is reduced to the solution of

$$\left( \frac{\partial^2}{\partial r^2} + r^{-1} \frac{\partial}{\partial r} + r^{-2} \frac{\partial^2}{\partial \theta^2} \right)^2 \phi = 0,$$

$0 < r < \infty$ ,  $-\alpha < \theta < \alpha$ , subject to the boundary conditions  $\phi_{rr}$  and  $(r^{-1} \phi_{\theta\theta})$ , assigned at  $\theta = \pm \alpha$ . This is done with the aid of the Mellin transform and particular forms of the boundary conditions are considered.

A. E. Heins.

Bellman, Richard. On the existence and boundedness of solutions of nonlinear partial differential equations of parabolic type. Trans. Amer. Math. Soc. 64, 21-44 (1948).

Let  $B$  be the surface of the cube  $R$  defined by  $0 < x, y, z < \pi$ , and let  $\|f\| = \max_R |f(x, y, z)|$  where  $f$  is assumed to be a continuous function. The principal object of the paper is to show that there exists a positive constant  $C$  such that if  $\|f\| \leq C$  then there is a solution  $u(x, y, z, t)$  of the nonlinear equation

$$(1) \quad u_{xx} + u_{yy} + u_{zz} - u_t = \sum_{i=2}^{\infty} d_i u^i, \\ (x, y, z) \in R, 0 < t < \infty,$$

satisfying the boundary conditions  $u = 0$ ,  $(x, y, z) \in B$ ,  $t > 0$ ,  $\lim_{t \rightarrow 0} u = f(x, y, z)$ ,  $(x, y, z) \in R$ , provided the series in the right member of (1) is absolutely convergent for  $|u| < \rho$ ,  $\rho > C$ . In previous attacks on this type of problem more stringent conditions were placed on the function  $f(x, y, z)$ . The later pages contain several misprints; however, a more troublesome point is to decide whether the  $d_i$  in (1) are constants or functions of the independent variables.

F. G. Dressel (Durham, N. C.).

Krzyżanowski, Miroslaw. Sur les solutions de l'équation linéaire du type parabolique déterminées par les conditions initiales. Ann. Soc. Polon. Math. 20 (1947), 7-9 (1948).

A function  $g(x_1, \dots, x_n, y) = g(x, y)$  will be said to belong to class  $E$  in  $0 \leq y \leq h$  if it is continuous and there exist two positive constants  $M$  and  $K$  such that

$$|g(x, y)| \leq M \exp(K \sum_{i=1}^n x_i^2).$$

Let  $R_S$  be the region  $0 \leq y \leq h$ ,  $-S < x_i < S$  ( $i = 1, \dots, n$ ), and let  $F(u)$  be the parabolic operator

$$\sum_{i,j=1}^n a_{ij}(x, y) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n a_i(x, y) \frac{\partial u}{\partial x_i} + a(x, y)u - \frac{\partial u}{\partial y},$$

where  $\sum a_{ij} \lambda_i \lambda_j$  is a positive definite form. The author feels that theorems II and III of his earlier paper [same Ann. 18, 145-156 (1945); these Rev. 8, 209] are inexact, and as partial replacement states the following weaker theorem. If  $a_{ij}$ ,  $a_i$ ,  $a$ ,  $f$  are continuous functions in  $R_S$ , such that  $|a_{ij}| < N$ ,  $|a_i|$ ,  $|a| < N(\sum |x_i| + 1)$  for some constant  $N$ ; and if for every continuous function  $\Phi(x, y)$  and every  $S$  there exists in  $R_S$  a solution of  $(*) F(u) = f(x, y)$  equal to  $\Phi$  on the boundary (excluding  $y=h$ ) of  $R_S$ , then for  $h$  sufficiently small and  $\psi(x)$  of class  $E$  there exists in  $R_S$  a solution  $u$  of  $(*)$  of class  $E$  such that  $u(x, 0) = \psi(x)$ .

F. G. Dressel.

Astolfo, Elvira. Valutazioni per eccesso della più bassa frequenza nelle oscillazioni proprie di una piastra ellittica omogenea incastrata. Ricerca Sci. 17, 1983-1986 (1947).

L'auteur se propose une détermination approximative de la première valeur propre  $\lambda_0$  de l'équation  $\Delta \Delta v - \lambda v = 0$ , sous condition que la fonction  $v$  s'annule sur le contour d'une ellipse, aussi bien que sa dérivée normale; on sait que, sous les mêmes conditions la solution de l'équation rend minima l'intégrale  $\iint_S (\Delta v)^2 dx dy$  étendue à l'aire de l'ellipse. Après changement des coordonnées cartesiennes en polaires, l'auteur suppose  $v$  développée en série de  $r^2$  et des cosinus des multiples de  $2\theta$  et considère comme approximations successives de  $v$  les expressions polynomiales formées par le premier, les deux premiers et les trois premiers termes du développement. Il en résulte une suite de problèmes de minimum algébriques d'où on déduit des valeurs numériques de  $\lambda_0$  qu'on considère comme des approximations successives de la valeur cherchée. Une table des résultats correspondants aux excentricités, de dixième en dixième, entre 0 et 0.9, permet de supposer que, du moins pour les petites excentricités, la méthode est satisfaisante.

B. Levi.

Mangeron, D. I. Recherches sur les problèmes à la frontière pour une classe d'équations aux dérivées partielles d'ordre supérieur. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iași] 2, 89-92 (1947).

On donne sans démonstration une suite de cinq théorèmes relatifs à la limitation par excès de la valeur maxime d'une

solution de l'équation

$$\partial^{2n}u/\partial x_1^{k_1} \cdots \partial x_n^{k_n} - A(x_1, \dots, x_n)u = f(x_1, \dots, x_n),$$

s'annulant sur la frontière d'un parallélépipède donné.

B. Levi (Rosario).

Hurgin, Ya. I. On the uniqueness of the solution of Cauchy's problem for linear partial differential equations. Izvestiya Akad. Nauk SSSR. Ser. Mat. 12, 213-223 (1948). (Russian)

The equation  $\sum_{k=0}^l a_k(t) \partial^k u / \partial t^k = Au$  is considered, where the  $a_k$  are continuous and  $A$  is the differential operator

$$A = \sum_{k_1 + \dots + k_n \leq m} a_{k_1, \dots, k_n}(x_1, \dots, x_n) \frac{\partial^{k_1 + \dots + k_n}}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}}.$$

The operator  $A$  is assumed to be continuous and the functions  $a_{k_1, \dots, k_n}$  sufficiently smooth so that a Green's formula may be proved for  $A$ . The equation may be of any type: elliptic, parabolic, etc. Let  $G_0$  be a bounded region in  $(x_1, \dots, x_n, t)$ -space containing the origin,  $F_0$  its boundary and  $G$  and  $F$  the parts of  $G_0$  and  $F_0$  for which  $t > 0$ . Then the author proves the following uniqueness theorem. If there exists a function  $u$  satisfying the differential equation in  $G$  such that, on  $F$ ,

$$u = 0, \quad \frac{\partial^p u}{\partial t^p} = 0 \quad (p < l),$$

$$\frac{\partial^q u}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} = 0 \quad (k_1 + \dots + k_n = q < m),$$

then  $u = 0$  in  $G$ .

The theorem is proved by considering  $A$  as an operator in a Hilbert space formed by certain functions over  $(x_1, \dots, x_n)$  which in particular have the property that they vanish to order  $m$  on a sphere sufficiently large to contain the intersection of  $G$  and the hyperplane  $t = t_0$  for any  $t_0$ . In this Hilbert space  $H$  may be extended to a hypermaximal operator defined on a certain subspace  $\Omega_A$ . The partial differential equation may then be regarded as an ordinary differential equation where  $u(t)$  is a function of  $t$  taking values in  $\Omega_A$ . For such an equation uniqueness may be proved by Picard's method if  $A$  is bounded. But with the aid of spectral decomposition  $\Omega_A$  may be considered as the sum of subspaces, on each of which  $A$  is bounded. Uniqueness is then proved for the projection of  $u$  on each such subspace and hence for  $u$  itself. The uniqueness theorem is also extended to the general case  $Lu = 0$ , where

$$Lu = \sum_{k_0 + k_1 + \dots + k_n \leq m} a_{k_0, k_1, \dots, k_n}(t) \frac{\partial^{k_0 + k_1 + \dots + k_n} u}{\partial t^{k_0} \partial x_1^{k_1} \cdots \partial x_n^{k_n}}.$$

P. A. Lagerstrom (Pasadena, Calif.).

Salehov, G. S. On Cauchy's problem for linear partial differential equations in the domain of infinitely differentiable functions. Uspehi Matem. Nauk (N.S.) 2, no. 2(18), 226-228 (1947). (Russian)

[Summary of a Kazan thesis, 1946.] It is known that for the analytic system of equations

$$\frac{\partial^{n_i} Z_i}{\partial t^{n_i}} = F_i \left( t, x_1, \dots, x_n; Z_1, \dots, Z_N; \dots; \frac{\partial^{k_0 + \dots + k_n} Z_k}{\partial t^{k_0} \partial x_1^{k_1} \cdots \partial x_n^{k_n}}; \dots \right),$$

$i = 1, 2, \dots, N$ , where  $n_i > k_0$ , in the "abnormal case" (when the condition  $n_i \geq \sum_{k=0}^l k$ , is not satisfied) the formal power series solution corresponding to certain analytic initial conditions need not converge [S. Kowalevsky, J. Reine Angew.

Math. 80, 1-32 (1875)]. C. Riquier showed [C. R. Acad. Sci. Paris 125, 1018-1019 (1897)], by examining particular examples, that the existence of analytic solutions of abnormal systems of linear partial differential equations depends not only on the nature of the initial data but also on the nature of the coefficients of the linear equations.

This thesis is concerned with Cauchy's problem for a system of linear partial differential equations, when the given initial data are only required to be infinitely differentiable functions. The main problems treated are the following. (1) What necessary and sufficient conditions must the infinitely differentiable functions  $\varphi_{ik_i}$ , having a common domain of definition in  $(x_1, \dots, x_n)$ -space, satisfy in order that the Cauchy initial value problem, consisting of the system of equations with constant coefficients

$$(1) \quad \frac{\partial^{p_i} Z_i}{\partial t^{p_i}} = \sum_{j=0}^N \sum_{\lambda_{ij}=0}^{r_{ij}} \sum_{\lambda_{ij} < 0} A_{ij} \varphi_{ik_1, \dots, k_n} \frac{\partial^{k_0 + k_1 + \dots + k_n} Z_j}{\partial t^{k_0} \partial x_1^{k_1} \cdots \partial x_n^{k_n}},$$

$i = 1, 2, \dots, N$ ;  $\lambda = \sum_{k=1}^l \lambda_k$ ,  $p_i > r_{ij}$ , and the initial conditions

$$\partial^{k_i} Z_i / \partial t^{k_i} \Big|_{t=0} = \varphi_{ik_i}(x_1, \dots, x_n),$$

$k_i = 0, 1, \dots, p_i - 1$ ;  $i = 1, 2, \dots, N$ , have a solution analytic in  $t$  in a neighborhood of  $t = 0$ ? (2) If there is such a solution, what is its behavior relative to the remaining variables  $x_1, \dots, x_n$ ? Necessary and sufficient conditions on the  $\varphi_{ik_i}$  are obtained in terms of the sequence of numbers  $\limsup |\varphi_{ik_i}^{(n)}(x_1, \dots, x_n)| \leq M_n$ , where  $n$  is the sum of the orders of the partial derivatives with respect to  $x_1, \dots, x_n$ . The same problems are investigated when the coefficients of the system (1) are themselves infinitely differentiable functions. Using the theory of entire, analytic, quasi-analytic and infinitely differentiable functions in the sense of Gevrey, the author solves the above problems for a broad class of systems of equations, irrespective of the normality or abnormality of the system.

Dealing first with a single equation, the problems are solved completely for the equations

$$(2) \quad \partial^p Z / \partial t^p \pm \partial^q Z / \partial x^q = 0,$$

obtaining as special cases known results of S. Kowalevsky, Le Roux, Holmgren, Goursat, and Gevrey for the heat equation. In the discussion of the system (1) the concept of the "weight"  $\delta$  of the system is found useful. By definition,  $\delta$  is the minimum of  $(p_i - r_{ij})/q_i$ ,  $i = 1, \dots, N$ , where, for each  $i$ ,  $r_{ij}$  is the maximum of the  $r_{ij}$  ( $j = 1, \dots, N$ ), and  $q_i$  is the maximum of the sums  $\sum_{k=1}^l \lambda_k$  for the given index  $i$ . The author shows that the weight of a system of equations serves to characterize the admissible initial data  $\varphi_{ik_i}$  and the integrals of the corresponding Cauchy problem. This leads to a classification of systems according to their weight. The customary distinction of "normal" and "abnormal" corresponds to weights greater than or equal to 1 and less than 1, respectively. The majorant series employed enable a determination of a nonlocal domain of convergence of the solutions, even in the domain of infinitely differentiable functions. The results obtained for (2) are extended to the general linear equation with constant coefficients, certain kinds of linear equations with infinitely differentiable coefficients and finally to systems of linear equations. The last part deals with weight preserving transformations of equations, and their application to the solution of certain equations. For example, in each case, for the appropriate equation (2), generalizations of d'Alembert's solution of the vibrating string equation and Poisson's integral for the heat equation are obtained. J. B. Diaz (Providence, R. I.).

## Integral Equations

Jacob, Caius. Sur une équation intégrale singulière. *Mathematica, Timișoara* 23, 153–156 (1948).

The author studies the singular integral equation

$$(1) \quad g(x) = \pi^{-1} \int_{-a}^a (x-t)^{-1} f(t) dt,$$

where the integral is a Cauchy principal value and  $g(x)$  is known. Using remarks which are function-theoretic in character, the author provides the solution of (1) in a fashion different from Söhngen [Math. Z. 45, 245–264 (1939)].

A. E. Heins (Pittsburgh, Pa.).

Sarmanov, O. V. On the rectification of a symmetric correlation. *Doklady Akad. Nauk SSSR (N.S.)* 58, 745–747 (1947). (Russian)

[This paper extends the results of a previous note, C. R. (Doklady) Acad. Sci. URSS (N.S.) 53, 773–776 (1946); these Rev. 8, 467.] Let  $F(x, y)$  be a symmetric distribution function for the variables  $(x, y)$ . If there exists a monotone solution of the equation

$$(1) \quad \varphi(x) = \lambda \int_{-\infty}^{\infty} \varphi(y) \{ F(x, y) / p(x) \} dy,$$

where  $p(x) = \int_{-\infty}^{\infty} F(x, y) dy$ , the change of variables  $X = \varphi(x)$ ,  $Y = \varphi(y)$  defines a new distribution function in which the correlation of  $X$  and  $Y$  is linear. In this case  $F(x, y)$  is called rectifiable. The author defines

$$F^{(k)}(x, y) = \int_{-\infty}^{\infty} \{ F^{(k-1)}(x, t) F(y, t) / p(t) \} dt$$

and requires that  $F(x, y) \{ p(x) p(y) \}^{-1}$  be of integrable square. Supposing that the smallest eigenvalue  $\lambda_1$  of (1) is of order  $m \geq 1$ , and denoting by  $\varphi_1, \dots, \varphi_m$  a corresponding orthogonal set of eigenfunctions, he defines

$$\begin{aligned} \Phi(x, y, h) &= \lim_{k \rightarrow \infty} \lambda_1^k \left\{ \frac{F^{(k)}(x+h, y)}{p(x+h)} - \frac{F^{(k)}(x, y)}{p(x)} \right\} \\ &= \{ \varphi_1(x+h) - \varphi_1(x) \} \varphi_1(y) p(y) + \dots \\ &\quad + \{ \varphi_m(x+h) - \varphi_m(x) \} \varphi_m(y) p(y). \end{aligned}$$

The following theorems are proved. (I) If there exists at least one value  $y = y_0$  for which the function

$$(2) \quad \Psi(x, y, h) = \int_{-\infty}^y \Phi(x, t, h) dt$$

is of constant sign for all  $x$  and all positive  $h$  (i.e.,  $h > 0$ ), then equation (1) admits a unique monotone solution. [Uniqueness was proved in the previous note.] (II) If  $\lambda_1$  is a simple root, then for monotonicity of the corresponding eigenfunction  $\varphi_1(x)$  it is necessary and sufficient that (2) be of constant sign for all  $x, y$ , and positive  $h$ . In general, if any one of the functions

$$\Psi_k(x, y) = \int_{-\infty}^y \frac{\partial}{\partial x} \frac{F^{(k)}(x, t)}{p(x)} dt, \quad k = 1, 2, \dots$$

is of constant sign for all  $x$  and  $y$ , (2) is of constant sign for all  $x, y$ , and positive  $h$  [cf. the author's previous note].

A. A. Brown (Cambridge, Mass.).

Solodovnikov, V. V. Criteria for the quality of a regulation. *Doklady Akad. Nauk SSSR (N.S.)* 60, 977–980 (1948). (Russian)

The author considers the function  $f(t)$  represented by the integral

$$(1) \quad f(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} z^{-1} J(z) e^{zt} dz,$$

where the path of integration is the imaginary axis indented to the right at the origin. This integral represents the solution of a differential equation treated by the author in a previous paper unavailable to the reviewer. From physical considerations the function  $f(t)$  is required to satisfy the conditions (2)  $0 \leq f(t) \leq g(t)$ ,  $0 \leq t \leq T_0$ ,  $g(t) \leq f(t) \leq g_1(t)$ ,  $t \geq T_0$ . Necessary and sufficient conditions on  $X(t)$  and  $Y(t)$  are given, where  $J(it) = X(t) + iY(t)$ , that (2) be satisfied.

R. Bellman (Stanford University, Calif.).

Davison, B. Large spherical hole in a slightly capturing medium. National Research Council of Canada. Division of Atomic Energy. Document no. MT-124 (N.R.C. 1552), i+39 pp. (4 plates) (1945).

The problem considered in the paper can be formulated as follows: to find a solution of the integral equation

$$\rho(r) = \frac{1-\alpha}{4\pi} \int \int \int \rho(r') R^{-3} e^{-(R-r)} dr', \quad |r| > a,$$

which has the asymptotic behavior

$$\rho(r) = (\pi/(2\nu r))^{\frac{1}{2}} \sum_{l=0}^{\infty} (2l+1) S_l(r/r) I_{l+1}(\nu r),$$

where  $\alpha (< 1)$  and  $a$  are two constants,  $R = |r-r'|$ ,  $\delta$  is the length of the chord intercepted by  $r-r'$  on a sphere of radius  $a$ ,  $S_l(r/r)$  is a spherical harmonic of order  $l$ ,  $I_{l+1}(\nu r)$  is the Bessel function for a purely imaginary argument and, finally,  $\alpha$  is the root ( $\alpha < 1$ ) of the transcendental equation  $1/(1-\alpha) = (2\nu)^{-1} \log(1+\nu)/(1-\nu)$ . If we express  $\rho(r)$  in the form  $\rho(r) = \sum_{l=0}^{\infty} (2l+1) S_l(r/r) n_l(r)$  it can be shown that

$$\begin{aligned} r n_l(r) &= \frac{1}{2} (1-\alpha) \int_a^{\infty} r' n_l(r') dr' \\ &\quad \times \int_{|r-r'|}^{r+r'} R^{-1} e^{-(R-r)} P_l \left( \frac{r^2+r'^2-R^2}{2r'} \right) dR. \end{aligned}$$

The case  $a \gg 1$  and  $\alpha \ll 1$  is considered. Writing  $r = a+x$ ,  $r' = a+y$  and  $r n_l(r) = f_l(x) + O(1/a)$  ( $x \ll a$ ), we find that  $f_l(x)$  satisfies the integral equation

$$f_l(x) = \frac{1}{2} \int_0^{\infty} f_l(y) [E(|x-y|) + \sum_k A_{l+k} E_{2k+1}(x+y)] dy,$$

where the  $E$ 's denote the exponential integrals and the  $A_{l+k}$ 's are the coefficients of  $\mu^k$  in the polynomial representation of  $P_l(1-2\mu^2)$ . The function  $f_l(x)$  can be interpreted as the density,  $\frac{1}{2} \int_{-1}^1 I_l(x, \mu) d\mu$ , in an isotropically scattering, nonabsorbing, semi-infinite atmosphere which satisfies the boundary condition  $I_l(0, -\mu) = I_l(0, \mu) P_l(1-2\mu^2)$  ( $\mu > 0$ ). This last circumstance allows us to conclude that  $f_l(x) = e^{ix} C^{(l)}(\nu a) [x + q_l(x)]$  ( $l > 0$ ) and  $f_0(x) = \text{constant} = f_{00}$ , where  $q_l(x)$  is bounded and nonnegative. A somewhat detailed consideration now shows that

$$C^{(l)}(\nu a) = (2l+1) / \sum_{k=0}^l \frac{(l+k)!}{k!(l-k)!(2\nu a)^k}$$

and  $f_{00} = e^{i\nu a} / (1+\nu a)$ .

In a first approximation the  $q_i$ 's may be treated as constants having the values  $q_1 = \frac{2}{3} + 2 \int_0^a \mu^2 P_1(1 - 2\mu^2) d\mu$ . The errors in this approximation and certain other related matters are also discussed.

S. Chandrasekhar.

**Davison, B.** Neutron density at the centre of a small spherical cavity. National Research Council of Canada. Division of Atomic Energy. Document no. MT-136 (N.R.C. 1555), i+26+7 pp. (1945).

One of the problems treated in this paper refers to the same integral equation as that considered in the preceding review except that now the restriction is that  $a$  is small; but  $a (< 1)$  is not restricted in any other way. Under these conditions it is sufficient to consider the equation for  $l=0$  and the problem reduces to finding a solution of the equation

$$(1) \quad rn(r) = \frac{1}{2}(1-a) \int_a^\infty r'n(r') dr' \int_{|r-r'|}^{r+r'} R^{-1} e^{-(R-B)} dR$$

which has the form

$$(2) \quad n(r) = S_0 \{ (nr)^{-1} \sinh nr + q_0(r) \}$$

( $S_0 = \text{constant}$ ), where  $q_0(r)$  tends to zero for any  $r$  as  $a$  tends to zero and, for any fixed  $a$ ,  $q_0(r) = O(e^{-nr})$  as  $r \rightarrow \infty$ . [The various symbols have the same meanings as in the preceding review.] The author shows that  $rq_0(r) = f_0(r; a) + O(a^3)$  ( $a \rightarrow 0$ ), where  $f_0(r; a)$  is the nonhomogeneous part of the integral equation for  $q_0(r)$  obtained by substituting (2) in (1).

The evaluation of the density at  $r=0$  according to

$$n_0(r) = (1-a) \int_a^\infty n(r') e^{-(r'-a)} dr'$$

is next carried out and shown to yield

$$(3) \quad n(0) = S_0 \{ 1 + a(a + \frac{1}{2}a^2) + \frac{1}{2}a(1-a)a^2(\frac{1}{4}\pi^2 - 1) + O(a^3 \log a) \}.$$

In physical terms the foregoing problem considers the effect of a small spherical cavity in an infinite, uniform, capturing medium when there are no sources at finite distances but an arbitrary distribution of sources (and sinks) at infinity. The author next shows that so far as effects of  $O(a)$  and  $O(a^2)$  are concerned, the perturbation in the density at the center of the cavity is also given by (3), in the case when there is a single point source situated right in the medium, at a distance  $r_0$  from the center of the hole, provided only  $r_0 \gg a$ . The case when there is in addition a small cavity surrounding the point source is also considered; it is then shown that in the same approximation as (3) the effects on the density, by the presence of the two holes, are additive.

S. Chandrasekhar (Williams Bay, Wis.).

**Davison, B.** Influence of an air gap surrounding a small black sphere upon the linear extrapolation length of the neutron density in the surrounding medium. National Research Council of Canada. Division of Atomic Energy. Document no. MT-232 (N.R.C. 1556), i+20 pp. (1946).

The integral equation appropriate for the problem stated in the title is

$$r\psi_0(r) = \frac{1}{2} \int_{a'}^\infty r'\psi_0(r') \{ E(|r-r'|) - E((r^2-b^2)^{\frac{1}{2}} + (r'^2-b^2)^{\frac{1}{2}}) \} dr' + \int_{(r^2-b^2)^{\frac{1}{2}} + (r'^2-b^2)^{\frac{1}{2}}}^{(r^2-b^2)^{\frac{1}{2}} + (r'^2-b^2)^{\frac{1}{2}}} R^{-1} e^{-R+B'} dR$$

( $r \geq b$ ), where  $b > a$ ,  $RR' = \{(R^2+r^2-r'^2)^{\frac{1}{2}} - 4R^2(r^2-b^2)^{\frac{1}{2}}\}^{\frac{1}{2}}$  and  $E$  denotes the exponential integral. The solution of this

equation is compared with that of the equation

$$r\psi_0(r) = \frac{1}{2} \int_{a'}^\infty r'\psi_0(r') \{ E(|r-r'|) - E(\sqrt{r^2-a^2} + \sqrt{r'^2-a'^2}) \} dr',$$

and it is shown that for  $a \ll 1$  the two solutions agree as  $r \rightarrow \infty$  provided  $a'$  is chosen in the following way:

$$a' = a + \frac{a^2}{6} \left\{ 1 - \frac{a}{b + (b^2 - a^2)^{\frac{1}{2}}} + \frac{a^3}{[b + (b^2 - a^2)^{\frac{1}{2}}]^2} \right\} + O(a^3 \log a).$$

S. Chandrasekhar (Williams Bay, Wis.).

**Platone, Giulio.** Sul passaggio da certe equazioni algebrico-funzionali a quelle integro-funzionali ed estensione di alcune proprietà fondamentali del nucleo risolvente generalizzato. *Pont. Acad. Sci. Acta* 9, 229-233 (1945). The author considers integro-functional equations of the type

$$\gamma(x, y; \lambda_1) - \gamma(x, y; \lambda_2) = (\lambda_1 - \lambda_2) \int_E A(\xi) \gamma(x, \xi; \lambda_1) \gamma(\xi, y; \lambda_2) d\xi$$

for a function  $\gamma(x, y; \lambda)$ , where  $x, y, \xi$  vary over a set  $E$ .

F. John (New York, N. Y.).

### Functional Analysis, Ergodic Theory

**Ogasawara, Tôzirô.** Compact metric Boolean algebras and vector lattices. *J. Sci. Hiroshima Univ. Ser. A* 11, 125-128 (1942).

The following theorems are proved. (1) Let  $L$  be any Banach lattice with unit element 1, such that the interval  $0 \leq x \leq 1$  is compact. Then  $L$  is strongly equivalent to a sequential vector space. (2) Let  $B$  be any Boolean algebra with metric  $\delta(x, y)$  derived from a sharply positive additive functional  $m(x)$  by  $\delta(x, y) = m(x \vee y) - m(x \wedge y)$ . Then compactness of  $B$  is equivalent to:  $B$  is (order) complete, continuous, and atomic with a countable basis. (3) Let  $C$  be any complemented modular lattice with metric  $\delta$  derived from a sharply positive modular functional. Then compactness of  $C$  is equivalent to:  $C$  is (order) complete, continuous, atomic with a countable basis, and any set of mutually perspective elements is finite, that is,  $C$  is the direct sum of a countable number of finite projective geometries, including 2-element Boolean algebras. (4) An  $n$ -dimensional Archimedean vector lattice is always isomorphic with  $R^n$ , the set of  $n$ -tuples of real numbers  $x = (x_1, \dots, x_n)$ , where  $x \leq y$  means  $x_i \leq y_i$  for all  $i$ .

Theorem (1) is deduced from theorem (2) by considering the Boolean algebra  $B$  of elements  $e$  such that  $e \wedge (1-e) = 0$ . This  $B$  is shown to be compact under a metric derived from a sharply positive additive functional  $m(e)$ . The existence of  $m(e)$  is shown to be a consequence of the Banach existence theorem for functionals on a Banach space.

I. Halperin (Kingston, Ont.).

**Vulih, B. Z.** The product in linear partially ordered spaces and its application to the theory of operations. I. *Mat. Sbornik N.S.* 22(64), 27-78 (1948). (Russian)

**Vulih, B. Z.** The product in linear partially ordered spaces and its application to the theory of operations. II. *Mat. Sbornik N.S.* 22(64), 267-317 (1948). (Russian)

The author presents a detailed exposition, with a number of extensions, of results previously announced [C. R.

(Doklady) Acad. Sci. URSS (N.S.) 26, 850-854, 855-859 (1940); 41, 142-144, 187-190 (1943); 52, 95-98, 383-386, 475-478 (1946); these Rev. 2, 221, 222; 6, 130; 8, 468; 9, 41]. Close connections exist between the theory expounded here and earlier results obtained by Freudenthal, Kantorovich, M. and S. Krein, and Kakutani [Freudenthal, Akad. Wetensch. Amsterdam, Proc. 39, 641-651 (1936); Kantorovich, Rec. Math. [Mat. Sbornik] N.S. 2(44), 121-168 (1937); 7(49), 209-284 (1940); M. and S. Krein, Rec. Math. [Mat. Sbornik] N.S. 13(55), 1-38 (1943); Kakutani, Ann. of Math. (2) 42, 523-537, 994-1024 (1941); these Rev. 2, 317; 6, 276; 2, 318; 3, 205]. Let  $\mathfrak{X}$  be a partially ordered linear space over the real numbers (which are denoted by  $R$  in the sequel) of type  $S_1$ : i.e., (I) for some  $x \neq 0$ ,  $0 < x$ ; (II)  $0 < x$  and  $0 < y$  imply  $0 < x+y$ ; (III) for every  $x \in \mathfrak{X}$ , there exists an  $x_1 \in \mathfrak{X}$  such that  $x_1 - x > 0$ ; (IV)  $0 < x$ ,  $\alpha \in R$ , and  $0 < \alpha$  imply  $0 < \alpha x$ ; (V) every subset of  $\mathfrak{X}$  bounded above admits a least upper bound. The author first proves a number of relations which obtain in all spaces of type  $S_1$ ; for example, if  $A$  is a bounded set in  $\mathfrak{X}$  and  $\alpha \in R$ , then  $\inf(\alpha, \sup_{x \in A} x) = \sup_{x \in A} \inf(\alpha, x)$ .

Next, a further restriction is placed upon the spaces considered. A positive element in  $\mathfrak{X}$ , which may be denoted by the symbol 1, is said to be a unit if  $\inf(x, 1) > 0$  for all positive  $x \in \mathfrak{X}$ . It is supposed that  $\mathfrak{X}$  contains a unit element and that this unit is fixed once and for all. An element  $\alpha \in \mathfrak{X}$  is said to be unitary if  $\inf(\alpha, 1-\alpha) = 0$ , and the set of all unitary elements in  $\mathfrak{X}$  is denoted by  $\mathfrak{E}(\mathfrak{X})$ . It is proved that unitary elements behave generally like projection operators in Hilbert space. For every  $x \in \mathfrak{X}$ , let  $e_x$  (the "characteristic" of  $x$ ) be the least element in  $\mathfrak{E}(\mathfrak{X})$  with the property that  $\inf(|x|, 1-e_x) = 0$ ; it is shown that  $e_x$  exists for all  $x \in \mathfrak{X}$  and that  $e_x = \sup_n \inf(n|x|, 1)$ . Various other formal properties of  $e_x$  are also established.

Next, for all  $x \geq 0$ , let  $S(x)$  be the set of all sums  $\sum_{i=1}^n \alpha_i e_{\alpha_i}$ , where the  $\alpha_i$  are nonnegative real numbers,  $e_{\alpha_i} \in \mathfrak{E}(\mathfrak{X})$ , and the entire sum is less than or equal to  $x$ . For  $x, y \in \mathfrak{X}$  and non-negative, consider the set  $B$  of all sums  $\sum_{i=1}^n \alpha_i e_{\alpha_i} \inf(e_{\alpha_i}, e_y)$ , where  $\sum_{i=1}^n \alpha_i e_{\alpha_i} \in S(x)$  and  $\sum_{i=1}^n \beta_i e_{\beta_i} \in S(y)$ . If  $B$  is bounded above, the element  $\sup_{z \in B} z$  is defined to be the product  $xy$  of the elements  $x$  and  $y$ . If  $B$  is unbounded, then the product  $xy$  does not exist. For  $x$  and  $y$  nonpositive, the product  $xy$  is defined as  $\sup(x, 0) \cdot \sup(y, 0) + (-1) \inf(x, 0) \cdot (-1) \inf(y, 0) - \sup(x, 0) \cdot (-1) \inf(y, 0) - (-1) \inf(x, 0) \cdot \sup(y, 0)$ , if all four products exist; otherwise  $xy$  does not exist. It is proved that  $xy = yx$  if either  $xy$  or  $yx$  exists; that  $x(y+z) = xy + xz$  if  $xy$  and  $xz$  exist; that  $x \cdot 1 = x$  and  $x \cdot 0 = 0$ ; that  $(xy)z = x(yz)$  if  $xy$ ,  $yz$ , and  $(xy)z$  exist; that  $xy = 0$  if and only if  $\inf(|x|, |y|) = 0$ .

The author next considers the existence of inverse elements, as follows. If for an  $x \in \mathfrak{X}$ , there exists  $y \in \mathfrak{X}$  such that  $e_x = e_y$ ,  $xy$  exists, and  $xy = e_x$ , then  $y$  is said to be inverse to  $x$  and is denoted by  $x^{-1}$ . It is immediate that  $e^{-1} = e$  for all  $e \in \mathfrak{E}(\mathfrak{X})$  (in particular,  $0^{-1} = 0$ ) and that  $(x^{-1})^{-1} = x$ . A number of the usual formal properties of inverses in commutative rings are established, and it is shown that the product is a continuous function of both variables in the topology of  $\sigma$ -convergence.

The next topic treated is that of spaces  $\mathfrak{X}$  which are rings under the operations  $x+y$  and  $xy$ . An element  $x \in \mathfrak{X}$  is said to be bounded if  $|x| \leq C$  for some  $C \in R$ ; the set of all such  $x$ 's for a given nonnegative  $C$  is called a segment in  $\mathfrak{X}$ , and the set of all segments is denoted by  $\mathfrak{X}_0$ . Then  $\mathfrak{X}_0$  is clearly a space of type  $S_1$ ; it is a ring; and it admits the norm  $\|x\| = \inf E[C, CxR, |x| \leq C]$ . Then, by results obtained

by a number of authors [see, for example, I. M. Gelfand, Rec. Math. [Mat. Sbornik] N.S. 9(51), 3-24 (1941); these Rev. 3, 51],  $\mathfrak{X}_0$  is identifiable with a ring of real-valued continuous functions defined on a certain compact Hausdorff space  $Y$ . [Reviewer's note:  $\mathfrak{X}_0$  is identifiable with the set of all real-valued continuous functions defined on  $Y$  if and only if it is complete in its norm; and if this is the case,  $Y$  must enjoy a very strong disconnectivity property, in view of axiom V satisfied by  $\mathfrak{X}_0$ . See also B. Z. Vulih, Doklady Akad. Nauk SSSR (N.S.) 58, 733-736 (1947); these Rev. 9, 290.] It is also proved that any space  $\mathfrak{X}$  can be imbedded in a ring  $\tilde{\mathfrak{X}}$ , by means of a construction due to Pinsker [C. R. (Doklady) Acad. Sci. URSS (N.S.) 21, 6-9 (1938); 22, 216-219 (1939)]. The author remarks also that every space of type  $S_1$  can be identified with a space of real-valued continuous functions on a compact Hausdorff space  $Y$ ; this fact makes the introduction of multiplication in  $\mathfrak{X}$  trivial. [Reviewer's note: very similar considerations apply, of course, to Banach spaces satisfying Kakutani's axioms for an  $M$ -space [loc. cit.]. Much of the interest of the present paper lies in the fact that multiplication is defined without recourse to a representation theory.]

A number of applications of the constructions outlined above are discussed, e.g., to  $L_p$ ,  $l_p$ , and the space of all countably infinite sequences of real numbers. In each case, the product turns out to be the ordinary point-wise product, and the inverse  $x^{-1}$  is the element which is 0 wherever  $x$  is 0 and is  $1/x$  wherever  $x \neq 0$ . The product and inverse are defined only where these constructions yield elements of the original space.

A final section of part I deals with the calculation of positive linear functionals in spaces  $\mathfrak{X}$  of type  $S_1$  which satisfy a certain countability restriction. Here it is remarked that the conjugate space  $\tilde{\mathfrak{X}}$  of  $\mathfrak{X}$  is of type  $S_1$  and that, if  $\tilde{\mathfrak{X}}$  contains a unit element  $\Pi$ , then every continuous positive linear functional  $\varphi(x)$  on  $\mathfrak{X}$  can be written in the form  $\Pi(xy)$ , where  $y$  lies in a certain subspace of  $\tilde{\mathfrak{X}}$ . Using this result, the author exhibits the general continuous linear functionals on  $L_p$  and  $l_p$ . [See S. Banach, Théorie des Opérations Linéaires, Warszawa, 1932, pp. 61 et seq.] It is shown also that  $\mathfrak{X}$  can be identified with a certain subset of  $\tilde{\mathfrak{X}}$  only when  $\mathfrak{X}$  is a space admitting an inner product.

Part II of the paper deals with mappings  $u$  of a space  $\mathfrak{X}$  into a space  $\mathfrak{Y}$  (where  $\mathfrak{X}$  and  $\mathfrak{Y}$  are both of type  $S_1$  and both contain unit elements) such that  $\inf(|x|, |y|) = 0$  implies that  $\inf(|u(x)|, |u(y)|) = 0$ . Such operations are called disjunctive. Most operations considered are in addition additive and continuous either in the  $\sigma$ -topology or  $t$ -topology. The first section deals with general properties of these operations. It is shown that a  $t$ -continuous operation  $u$  is multiplicative (i.e.,  $x_1 x_2$  exists implies that  $u(x_1)u(x_2)$  exists and is equal to  $u(x_1 x_2)$ ) if and only if  $u(\mathfrak{E}(\mathfrak{X})) \subset \mathfrak{E}(\mathfrak{Y})$ . Relations existing among the properties of disjunctivity, multiplicativity, additivity, and  $t$ - and  $\sigma$ -continuity are investigated in detail. It is proved that an additive, multiplicative,  $\sigma$ -continuous mapping of  $\mathfrak{X}$  into itself has the form  $x \rightarrow xe$  for some  $e \in \mathfrak{E}(\mathfrak{X})$  if and only if  $x \geq 0$  implies  $u(x) \leq x$ . Other formal properties of multiplicative operations are discussed, among them being the theorem that a one-to-one  $\sigma$ -continuous additive and multiplicative operation admits an inverse having all of these properties.

It is next proved that all  $\sigma$ -continuous additive and multiplicative operations  $u$  mapping  $M$  (the space of all bounded measurable functions on  $[a, b]$ ) into itself have a particularly simple form, namely,  $u(x) = x(\varphi(s))$  for  $s \in E$ ;  $u(x) = 0$

for  $s$  none, where  $E$  is a measurable subset of  $[a, b]$  and  $\varphi$  is a measurable function mapping  $[a, b]$  into itself, with the property that the inverse image of a set of measure 0 has measure 0. A similar result is proved for mappings of  $L_1$  into itself.

The author then exhibits a theory of the Radon integral for spaces  $\mathfrak{X}$  of type  $S_s$  having a unit element. A spectral resolution for  $xR$  is first defined:  $e_\lambda$  ( $\lambda \in R$ ) is the characteristic element for  $\inf(x - \lambda I, 0)$ . Let  $\varphi$  be a function defined on  $\mathfrak{E}(\mathfrak{X})$  and with values in  $\mathfrak{E}(\mathfrak{Y})$ , where  $\mathfrak{Y}$  is another space of type  $S_s$  with unit element. For  $e_1, e_2 \in \mathfrak{E}(\mathfrak{X})$  and  $\inf(e_1, e_2) = 0$ , suppose that  $\varphi(e_1 + e_2) = \varphi(e_1) + \varphi(e_2)$ . Let  $y$  be a function with domain included in  $R$  and range included in  $\mathfrak{Y}$ . The symbol  $\int_{-\infty}^{\infty} y(\lambda) d\varphi(e_\lambda) x$  is defined as the  $\sigma$ -limit of sums  $\sum_{n=1}^{\infty} y(\lambda_n) \varphi(e_{\lambda_n} - e_{\lambda_{n-1}})$  (when this limit exists), where  $-\infty < \dots < \lambda_n < \dots < \lambda_1 < \dots < \lambda_n < \dots < +\infty$ , and  $\lambda_{n-1} \leq \lambda_n < \lambda_n$ , as  $\sup \lambda_n - \lambda_{n-1} \rightarrow \infty$ . For  $y(\lambda) = \lambda \cdot 1$ , many formal properties of this integral are described, and it is proved that any  $\sigma$ -continuous multiplicative additive operation mapping  $\mathfrak{X}$  into  $\mathfrak{Y}$  has a representation as an integral of this kind, and conversely. A similar representation, using more general functions  $y(\lambda)$ , is produced for operations  $u$  mapping  $\mathfrak{X}_0$  into  $\mathfrak{Y}$ . Such an operation  $u(x)$  has the form  $\int_{-\infty}^{\infty} y(\lambda) d(u(e_\lambda)) x$  if  $u$  is additive for disjoint elements, is continuous in a certain strong sense, and is disjunctive. The author closes with the exposition of a theory of approximation to operations of general type by means of  $\sigma$ -continuous additive and multiplicative operations, which reduces to the classical Weierstrass approximation theorem for the case  $X = Y = R$ .

It would be useful, in the reviewer's opinion, to investigate the theory when axiom V is relaxed in various ways, since this axiom is so stringent that it rules out many of the commonest partially ordered linear spaces.

E. Hewitt (Seattle, Wash.).

**Cohen, I. S. On non-Archimedean normed spaces.** Nederl. Akad. Wetensch., Proc. 51, 693-698 = Indagationes Math. 10, 244-249 (1948).

Let  $K$  be a field complete in a non-Archimedean real-valued valuation. The author considers normed linear spaces over  $K$  which satisfy, besides the standard axioms, the strong triangle inequality. Such spaces have been studied by Monna [same Proc. 49, 1045-1055, 1056-1062, 1134-1141, 1142-1152 = Indagationes Math. 8, 643-653, 654-660, 682-689, 690-700 (1946); these Rev. 9, 43], and the author simplifies and extends Monna's work. Finite-dimensional spaces are shown to carry the Cartesian product topology, and locally compact spaces are finite-dimensional. The Hahn-Banach theorem is proved under the assumption that the valuation of  $K$  is discrete, and a counter-example shows that it may fail otherwise.

I. Kaplansky.

**Karlin, S. Unconditional convergence in Banach spaces.** Bull. Amer. Math. Soc. 54, 148-152 (1948).

A series of elements  $x_i$  belonging to a weakly complete Banach space is said to be unconditionally convergent if for every variation of sign  $\epsilon_i = \pm 1$ ,  $\sum \epsilon_i x_i$  is convergent. The author calls  $\sum x_i$  unconditionally summable if there exists a row-finite Toeplitz matrix  $(b_{ij})$  such that for every variation of sign  $\epsilon_i = \sum_{j=1}^q b_{ij} \sum_{i=1}^q \epsilon_i x_j$  converges. The author shows that unconditional summability implies unconditional convergence. It is likewise shown that if  $\|\sum_{i=1}^q b_{ij} \sum_{i=1}^q \epsilon_i x_j\| \leq C$  for almost all variations of signs, then  $\sum x_i$  converges unconditionally. These results are then

applied to a complete bounded orthogonal system for which the Lebesgue kernel is summable by a row-finite Toeplitz matrix. It is shown that if, for every variation of sign,  $(\epsilon_i a_i)$  are the Fourier coefficients with respect to the orthogonal system  $\{\phi_k(t)\}$  of a function in  $L^p$  ( $p \geq 1$ ), then  $(d_i a_i)$  are likewise the Fourier coefficients of a function in  $L^p$  for every sequence of numbers  $d_i$  with  $|d_i| \leq 1$ .

R. S. Phillips (Los Angeles, Calif.).

**Shimizu, Tatsujiro. Analytic operations and analytic operational equations.** I. Math. Japonicae 1, 36-40 (1948).

The paper deals with the inversion of special types of power series in complex Banach spaces, namely  $y = x + U(x) + F(x)$ , where  $U(x)$  is a completely continuous linear operation and  $F(x)$  is a power series of homogeneous polynomials of degree 2, 3, ..., regular in a neighborhood of the origin. Reference is made to the theory of Riesz and Schauder for the linear case  $F(x) = 0$ . The general case is then treated as a straightforward abstraction of Schmidt's solution of nonlinear integral equations. A. E. Taylor (Los Angeles, Calif.).

**Sebastião e Silva, José. L'analisi funzionale lineare nel campo delle funzioni analitiche.** Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. (8) 1, 207-240 (1947).

This is a detailed exposition of the results announced in Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 709-715 (1946); these Rev. 8, 278. T. H. Hildebrandt.

\***Vigier, Jean-Pierre. Étude sur les Suites Infinies d'Opérateurs Hermitiens.** Thesis, University of Geneva, 1946. 35 pp.

A Hermitian operator  $A$  is a linear operator such that  $(Af, g) = (f, Ag)$ ; the domain is not required to be dense. A number of theorems are established initially concerning sequences of uniformly bounded operators and Hermitian operators with certain definiteness properties. For an individual Hermitian operator  $A$ , the iterates relative to a fixed  $x_0$  are defined as follows. Let  $l_1 = \|Ax_0\|$ ,  $x_1 = Ax_0/l_1$ ,  $l_2 = \|Ax_1\|$ ,  $x_2 = Ax_1/l_2$ . Let  $l = \lim l_n$ ,  $\omega(x_0) = (l_1/l)(l_2/l) \dots$ . Suppose now a sequence  $\{A^{(n)}\}$  of uniformly bounded Hermitian operators converges strongly to an operator  $A$ . It is shown that a necessary and sufficient condition that the iterates  $x_n$  relative to  $A$  should converge is that  $\limsup \omega^{(n)} \neq 0$ . An operator  $A$  is said to be regular if  $\omega(x_0) \neq 0$  if  $x_0 \neq 0$ . This condition implies the existence of a complete set of characteristic elements. It is shown that every bounded Hermitian operator is the limit of a sequence of regular operators and a variety of results equivalent to the spectral resolution are based on this fact.

F. J. Murray (New York, N. Y.).

**Julia, Gaston. Sur des systèmes de vecteurs généralisant les systèmes orthonormaux.** C. R. Acad. Sci. Paris 227, 168-170 (1948).

A characterization, in terms of the norms of their linear combinations, of those systems of  $q$  vectors in an  $n$ -dimensional vector space ( $q > n$ ) which may be obtained by projecting a basis in a  $q$ -dimensional space. P. R. Halmos.

**Hailov, Z. I. Linear singular equations in a normed ring.** Doklady Akad. Nauk SSSR (N.S.) 60, 1133-1136 (1948). (Russian)

Let  $R$  be a normed ring and let  $T$  be a linear operator on  $R$ . For arbitrary  $xR$  denote by  $T(x)$  the result of  $T$  operating on  $x$  and by  $xT, Tx$  the products of  $T$  with  $x$  considered as a linear operator on  $R$  by left multiplication.

Let  $F$  be a class of linear operators on  $R$  with the properties: (1) for each  $T \in F$ , the Riesz-Schauder theory [J. Schauder, *Studia Math.* 2, 1-6 (1930)] applies to  $E+T$  ( $E$  the identity operator); (2) if  $x \in R$  and  $T, T_1 \in F$ , then  $xT, Tx, T+T_1, TT_1 \in F$ . Let  $S$  be a linear operator on  $R$  with the properties: (i)  $S^2 = E$ ; (ii) if  $x \in R$ , then  $Sx - xS \in F$ ; (iii) if  $T \in F$ , then  $ST, TS \in F$ . It follows that  $Sx(y) = xS(y) + T(y)$  and  $SxS(y) = xy + T_1(y)$ , where  $T, T_1 \in F$ .

Consider the "singular" equation

$$(*) \quad K(x) = ux + vS(x) + T(x) = y,$$

where  $u, v, y \in R$  and the elements  $u \pm v$  are regular in  $R$ . If  $v=0$ , then  $(*)$  reduces to a Riesz-Schauder equation. If  $K_1(x) = ux + v_1S(x) + T_1(x)$ , where  $T_1 \in F$ , then  $K_1K, K_1K$  are of type  $K$ . In particular, if  $u_0$  is regular in  $R$  and if  $u_1 = \frac{1}{2}u_0\{(u+v)^{-1} + (u-v)^{-1}\}$ ,  $v_1 = \frac{1}{2}u_0\{(u+v)^{-1} - (u-v)^{-1}\}$ , then  $K_1K$  reduces to  $u_0E + T_1$ , where  $T_1 \in F$ . Let  $\bar{K}(X) = \bar{u}(X) + \bar{S}\bar{v}(X) + \bar{T}(X)$ , where  $X$  is in the conjugate space  $\bar{R}$  of bounded linear functionals on  $R$  and  $\bar{u}, \bar{v}, \bar{S}, \bar{T}$  are operators on  $\bar{R}$  adjoint respectively to  $u, v, S, T$ . The class  $\bar{F}$  of all  $T$  where  $T \in F$  and operators  $\bar{S}, \bar{T}$  have the same properties as  $F, S, K$  if  $x$  is replaced by  $\bar{x}$ .

The following theorems, which generalize known results for integral equations, are obtained. (I) The number of linearly independent solutions of  $K(x)=0$  and of  $\bar{K}(X)=0$  is finite. (II) In order for solutions of  $K(x)=y$  (respectively,  $\bar{K}(X)=Y$ ) to exist, it is necessary and sufficient that  $X_0(y)=0$  for every solution  $X_0$  of  $\bar{K}(X)=0$  (respectively,  $Y_0(x)=0$  for every solution  $x_0$  of  $K(x)=0$ ). (III) The number of linearly independent solutions of  $K(x)=0$  minus the number of linearly independent solutions of  $\bar{K}(X)=0$  depends only on  $u, v, S$ .

C. E. Rickart.

**Ditkin, V. A. Certain formulas for noncommutative operators.** *Uspehi Matem. Nauk (N.S.)* 3, no. 2(24), 234-237 (1948). (Russian)

Let  $R$  be a ring with two generators  $p$  and  $q$  and let  $A$  be a linear differentiation operator on  $R$  (i.e.,  $A(fg) = (Af)g + f(Ag)$ ). The author establishes a number of formulas of which a typical one is

$$f(p)q - qf(p) = \sum_{k=1}^{\infty} (-1)^{k-1} s_{k-1} f^{(k)}(p) / k!;$$

here  $f$  is a polynomial,  $s_0 = Ap$ , and  $s_k = A_p^k$  for  $k > 0$ , where  $A_p f = fp - pf$ . From these the author deduces another formula which when applied to operators for which  $pq - qp$  is a constant times the identity yields a generalization of a formula of McCoy [Trans. Amer. Math. Soc. 31, 793-806 (1929), in particular, p. 801]. There appears to be a lack of rigor in connection with this last formula, for exponentials of operators are used without being explicitly defined.

I. E. Segal (Chicago, Ill.).

**Barbašin, E. A. On dynamical systems with a velocity potential.** *Doklady Akad. Nauk SSSR (N.S.)* 61, 185-187 (1948). (Russian)

Let  $M$  be a locally Euclidean manifold with local coordinates  $x_1, \dots, x_n$  and let  $f(p, t)$  ( $p \in M, -\infty < t < \infty, f(p, t) \in M$ ) be the dynamical system (one-parameter transformation group) defined on  $M$  by a system of differential equations  $dx_i/dt = X_i(x_1, \dots, x_n)$  ( $i = 1, \dots, n$ ). The dynamical system is called linear when it is topologically equivalent to the translation group of a family of parallel straight lines in some Euclidean space. A point  $p$  of  $M$  is called wandering in case there exist a neighborhood  $S$  of  $p$  and

a number  $N$  such that  $f(S, t) \cap S = \emptyset$  for all  $t > N$ . The following theorems are stated and briefly discussed: (1) if there exists a point-function  $u$  of class  $C^1$  on  $M$  such that  $\sum_{i=1}^n (\partial u / \partial x_i) X_i \geq k > 0$  for some constant  $k$ , then the dynamical system is linear; (2) if there exists a velocity potential (a point-function  $u$  of class  $C^1$  on  $M$  such that  $X_i = \partial u / \partial x_i$  ( $i = 1, \dots, n$ )), then each point of  $M$  is either wandering or fixed.

W. H. Gottschalk (Philadelphia, Pa.).

**Barbašin, E. A. On homomorphisms of dynamical systems.** *Doklady Akad. Nauk SSSR (N.S.)* 61, 429-432 (1948). (Russian)

Let  $R$  be a locally compact metric space, let  $G$  be an Abelian topological group, both satisfying the second axiom of countability, and let  $f: R \times G \rightarrow R$  define a transformation group, denoted by  $(R, G)$ . A point  $x \in R$  is called stable in case its orbit  $f(x, G)$  has compact closure. It is assumed that at least one point of  $R$  is stable. Let  $K$  denote the circle group. A homomorphism of  $(R, G)$  is defined to be a mapping  $\alpha: R \rightarrow K$  such that  $\alpha(f(x, G)) = \alpha(x) + \alpha^*(g)$  ( $x \in R, g \in G$ ) for some character  $\alpha^*$  of  $G$ ; in case  $\alpha^* = 0$  the homomorphism  $\alpha$  is called invariant. Let  $L$  be the topological group of all homomorphisms of  $(R, G)$  with the obvious operation of addition and the compact-open topology and let  $H$  be the subgroup of all invariant homomorphisms;  $H$  is open and closed in  $L$ . The quotient group  $\Delta = L/H$  is called the character group of  $(R, G)$ ;  $\Delta$  is discrete and countable. The system  $(R, G)$  is called indecomposable in case each element of  $H$  is constant on  $R$ . Let  $A$  be the character group of  $\Delta$ . If  $g \in G$ , define  $a_g \in A$  as follows:  $a_g(\alpha + H) = \alpha^*(g)$  ( $\alpha + H \in \Delta, \alpha^*$  is defined as above). Let  $A_0$  denote the set of all  $a_g$  ( $g \in G$ );  $A_0$  is dense in  $A$ . Let  $(A, A_0)$  denote the transformation group defined by translating the elements of  $A$  by the elements of  $A_0$ .

The following theorems are stated. (1) If  $(R, G)$  is indecomposable, then  $L$  is the direct sum of  $K$  and  $\Delta$ . (2) The group  $\Delta$ , the character group of  $(A, A_0)$  and the character group of  $A$  are all isomorphic. (3) For each stable point  $x_0 \in R$  there exists a mapping  $\psi$  of  $R$  onto  $A$  such that: (a)  $\psi(f(x, g)) = a_g + \psi(x)$  ( $x \in R, g \in G$ ); (b)  $\psi(f(x_0, G)) = A$ ; (c) if  $R$  is indecomposable and if  $L$  separates  $x_1, x_2 \in R$ , then  $\psi(x_1) \neq \psi(x_2)$ . (4) The space  $R$  is a minimal orbit-closure and  $G$  is equicontinuous if and only if  $(R, G)$  is indecomposable and  $L$  is separating. (5) If  $L$  is separating, then  $G$  is equicontinuous on each compact orbit-closure. Some proofs are given. W. H. Gottschalk (Philadelphia, Pa.).

**Seifert, H. Periodische Bewegungen mechanischer Systeme.** *Math. Z.* 51, 197-216 (1948).

Let  $U(x_1, \dots, x_n) = U(x)$  be real-valued and analytic in a region  $A$  in  $R^n$  (Euclidean  $n$ -dimensional space) and let the subset  $B$  of points of  $A$  which satisfy  $U(x) \leq E$ ,  $E$  some fixed real number, constitute a homeomorph of the set  $\sum_{i=1}^n x_i^2 \leq 1$ . Furthermore, if  $I(B)$  denotes the interior of  $B$ , it is supposed that  $U(x) < E$  on  $I(B)$  and  $\text{grad } U \neq 0$  on  $F(B)$  ( $= B - I(B)$ ). Let  $\sum_{i,j=1}^n a_{ij}(x) dx_i dx_j$  be a positive definite symmetric quadratic form with coefficients analytic in  $A$ . There is thereby defined a dynamical system with Lagrangian coordinates  $(x_1, \dots, x_n)$ , with potential energy  $U$  and kinetic energy  $T(x, \dot{x}) = \sum_{i,j=1}^n a_{ij}(x) \dot{x}_i \dot{x}_j$ . It is proved that this dynamical system possesses a periodic motion with total energy  $E$ , a point of this motion moving back and forth on an arc with end-points on  $F(B)$ .

The problem can be considered as a geodesic problem. The behavior of the motions in the neighborhood of  $F(B)$

is analyzed and by suitable modification of the metric in the neighborhood of  $F(B)$  it is possible to obtain surfaces closed to  $F(B)$  and interior to  $B$ , which are geodesically convex. This permits use of a technique developed for use in the case of closed convex surfaces by G. D. Birkhoff [Dynamical Systems, Amer. Math. Soc. Colloquium Publ., v. 9, New York, 1927] whereby a family of curves is continuously deformed and a limiting geodesic arc is obtained.

G. A. Hedlund (New Haven, Conn.).

### Mathematical Statistics

Chakrabarti, M. C. On the inadequacy of measuring the peakedness of a distribution curve by the standardised fourth moment. *Bull. Calcutta Math. Soc.* 39, 154-156 (1947).

For some special distributions, it is shown that under certain conditions the measure of kurtosis,  $\beta_2 - 3$ , does not correctly indicate the peakedness of these distributions as compared to the normal distribution. R. L. Anderson.

Michalup, Eric. The characteristics. *Estados Unidos de Venezuela. Bol. Acad. Ci. Fis. Mat. Nat.* 11, 448-478 (1948). (Spanish)

An expository article on the measures of skewness and excess for statistical distributions. W. Kozakiewicz.

Boldrini, Marcello. Sulla teoria della media tipica. *Pont. Acad. Sci. Comment.* 10, 1-41 (1946).

Fréchet, Maurice. Le coefficient de connexion statistique de Gini-Salvemini. *Mathematica, Timisoara* 23, 46-51 (1948).

The author shows that Gini's "index of connection"  $g_{xy}$ , in the form to which it was transformed by Salvemini, satisfies six conditions which seem reasonable requirements of every quantity that may be considered a measure of correlation between two stochastic variables.

C. C. Craig (Ann Arbor, Mich.).

Friede, G., und Münzner, H. Zur Maximalkorrelation. *Z. Angew. Math. Mech.* 28, 158-160 (1948).

The greatest correlation between  $f(x)$  and  $g(y)$ , where  $f$  and  $g$  are arbitrary functions, was proposed by Hirschfeld [Proc. Cambridge Philos. Soc. 31, 520-524 (1935)] and Gebelein [same Z. 21, 364-379 (1941); 22, 171-173 (1942); these Rev. 4, 104, 279] as an invariant measure of correlation. The authors point out that it can be unity (e.g.,  $x$  and  $y$  integers both even or both odd together) when the regression lines are those of independent quantities. This does not affect the work of Bopp [Z. Naturforschung 2a, 202-216 (1947); these Rev. 10, 92] on the connection between the maximal correlation and quantum mechanics.

J. W. Tukey (Princeton, N. J.).

Wishart, John. Tests of significance in the simple regression problem. *J. Inst. Actuaries Students' Soc.* 8, 38-43 (1948).

Given  $(x_i, y_i)$ ,  $i=1, \dots, p$ , with the  $y_i$  distributed independently in a standardized normal probability function, the author derives by means of orthogonal transformations the well-known distributions of  $a - a$  and  $b - \beta$ , where  $a$  and

$b$  are least squares estimates of  $\alpha$  and  $\beta$  in the regression line  $y = \alpha + \beta(x - \bar{x})$ , and  $\bar{x}$  is the mean of the  $x_i$ .

L. A. Aroian (New York, N. Y.).

Hartley, H. O. The estimation of non-linear parameters by 'internal least squares.' *Biometrika* 35, 32-45 (1948).

The author observes that many of the nonlinear regression laws between two variables which have importance in applications arise from linear relationships between the dependent variable and its first and second derivatives. By substituting a difference equation which generates the same regression relation and summing it over the observed values he secures a linear "internal regression" equation from which the parameters can now be estimated by least squares. The method is illustrated in detail for the exponential law of diminishing returns. It is noted that the logistic curve may be reduced to this form as well as the Makeham-Gompertz curve and the first of these is used in a further numerical illustration. The special case of the exponential law first considered, in which the asymptote is known, is used for an investigation of the relative efficiency of the proposed method. The conclusion is that for large samples the precision of estimation tends to that given by the method of maximum likelihood. C. C. Craig (Ann Arbor, Mich.).

Cochran, W. G., and Bliss, C. L. Discriminant functions with covariance. *Ann. Math. Statistics* 19, 151-176 (1948).

The authors are concerned with the extension of the method of linear discriminant functions to a situation in which besides measurements on variables whose expected values are presumed different in the different populations sampled the experimenter also has measurements on other variables whose expected values in those universes are known to be the same. These latter variables have no discriminating power of themselves but individual members of the first set, the discriminators, are related to them by linear regression equations and they are the covariance variables. The general problem is the proper utilization of these covariance variables in view of the existence of these regressions in the discriminant analysis. The general plan is, fairly naturally, to adjust the discriminators by means of the "within sample" regressions, then calculate the discriminant functions, using the adjusted values of the discriminators in the usual way, and then apply the usual significance tests, appropriately modified. Preceding the theoretical investigation an actual numerical example is worked out to illustrate the form of the computations needed, which are presented as much in the terms of ordinary multiple regression calculations as possible. In the theoretical discussion the theoretically equivalent approach of Mahalanobis's generalized distance is used and from this an alternate computational scheme emerges. These developments, for the case of two populations, are carried on to derive the necessary sampling distributions and the appropriate significance tests from them. An illustrative case of more than two populations is also included to the extent of indicating how the results turn out without giving the intervening details. C. C. Craig (Ann Arbor, Mich.).

Vajda, S. A note on the use of weighted orthogonal functions in statistical analysis. *Proc. Cambridge Philos. Soc.* 44, 588-590 (1948).

If given values of  $y_{x_1, \dots, x_n}$ , with  $x_i = 1, \dots, n_i$ , are samples from normal populations with variances  $1/w_{x_1, \dots, x_n}$ , then a

statistical significance test can be based on the minimum of

$$\sum_{x_1} \cdots \sum_{x_n} w_{x_1 \cdots x_n} [y_{x_1 \cdots x_n} - \sum_{m_1 \cdots m_n} a_{m_1 \cdots m_n} Q_{m_1 \cdots m_n; x_1 \cdots x_n}]^2,$$

where the  $Q$ 's satisfy the condition that, if  $m_i = 0$ , then the value of  $Q$  is independent of  $x_i$ , and  $Q_{0 \cdots 0} = 1$ . Interactions and effects can be defined by the minimal sums of squares resulting when appropriate sets of  $a$ 's are omitted. In the special case of populations of unit variance, the  $Q$ 's may be taken as products in the form  $Q_{m_1 \cdots m_n; x_1 \cdots x_n} = p_{m_1 x_1} \cdots p_{m_n x_n}$ , where the  $p_{m_i x_i}$  form an orthogonal matrix. The author shows that only if the weights are proportional is there a set of  $Q$ 's leading to simple computations. In other cases the  $Q$ 's are not unique and a given simple choice may lead to irrelevant computations.

A. A. Bennett.

Nandi, H. K. A note on Student's  $t$  for paired samples.

Bull. Calcutta Math. Soc. 39, 61-64 (1947).

Let  $(x_1, x_2), (z_{1i}, z_{2i})$  ( $i = 1, \dots, n$ ) be  $n+1$  pairs of random samples from a bivariate normal population with variances  $\sigma_1^2, \sigma_2^2$ , the coefficient of correlation  $\rho$ , and  $E(z_{1i}) = E(z_{2i}) = 0, E(x_1) = m_1, E(x_2) = m_2$ . Put

$$ns_{kl} = \sum_i z_{ki} z_{li}, \quad k, l = 1, 2,$$

$$t = (x_1 - x_2) / (s_{11} + s_{22} - 2s_{12})^{1/2}, \quad W = (x_1 - x_2) / (s_{11} + s_{22})^{1/2}.$$

Two methods of testing for the equality of the means are considered. They consist in referring  $t$  and  $W$  to Student's distribution with  $n$  and  $2n$  degrees of freedom, respectively. It is found that: (a) the first test is an exact test; the second test is only an approximate one whose exact distribution involves the correlation coefficient  $\rho$ ; (b) for only positive values of  $\rho$  the first kind of error is controlled by the second test, this error being less than the level of significance; (c) for only small values of  $\rho$  (for example, for samples of five  $\rho$  must be less than 0.1) the second test shows an advantage over the first test in controlling the second kind of error. W. Kozakiewics (Saskatoon, Sask.).

David, F. N., and Johnson, N. L. The probability integral transformation when parameters are estimated from the sample. Biometrika 35, 182-190 (1948).

It is well known that the probability integral transformation enables one to replace a set of  $n$  independent variables  $x_i$  obeying a common continuous distribution law by a set of independent variables  $y_i$  which are rectangularly distributed. The authors observe that this is possible only if the distribution law for  $x$  is completely specified as to its parameters and they investigate the consequences of estimating these parameters as functions of the sample  $x_i$ . They show that in this case the  $y_i$  are in general no longer independent and obey a distribution law which depends on the functional form of the common probability law  $f(x)$  of the  $x$ 's. As applications they consider the special case in which  $f(x)$  contains two parameters only, one for location and one for scaling, and derive a condition that the  $y_i$  be rectangularly distributed, which they note usually will not be satisfied. The case in which  $f(x)$  is the normal distribution function is studied in detail when the mean alone or both the mean and variance are estimated in the usual way and a like detailed study is carried out for a  $\chi^2$ -distribution with 2 degrees of freedom containing a single parameter. In both cases the joint distribution of the  $y$ 's is obtained.

C. C. Craig (Ann Arbor, Mich.).

Johnson, N. L. Tests of significance in the variate difference method. Biometrika 35, 206-209 (1948).

The author studies the problem of finding a test of the hypothesis that a time series  $u_t, t = 1, \dots, n$ , can be assumed to be of the form  $u_t = f_t + z_t$  (where  $z_t$  are mutually independent (normally distributed) random residuals with expected value zero and standard deviation  $\sigma$  and  $f_t$  is a smooth function with differences (of order  $k$ )  $\Delta^k f_t = 0$  for  $k \geq K$ ), more effective than the test given by G. Tintner [The Variate Difference Method, Cowles Commission Monograph no. 5, 1940; these Rev. 1, 250]. Tintner uses the variance ratio

$$F = \frac{\sum_r (\Delta^k u_r)^2 \cdot 2(2k+1)}{\sum_s (\Delta^{k+1} u_s)^2 (k+1)},$$

where the differences  $\Delta^k u_r$  and  $\Delta^{k+1} u_s$  are selected in such a way, for instance  $s_i = r_i + k + 1, r_i = t_0 + i \cdot (2k+3); i = 0, 1, \dots, j-1$ , that they would be mutually uncorrelated for  $k \geq K$  if the hypothesis were true. The test criterion  $F$  is (in the case when the residuals are mutually independently normally distributed) distributed as Snedecor's (Fisher's?)  $F$  with  $(j, j)$  degrees of freedom. Johnson uses a result due to Hsu [Biometrika 31, 221-237 (1940); these Rev. 2, 111] and finds that the test criterion

$$L = \frac{T_{k,k} T_{k+1,k+1} - T_{k,k+1}^2}{(2k+1) \{ T_{kk} + \frac{1}{2}(k+1)(2k+1)^{-1} T_{k+1,k+1} + T_{k,k+1} \}^2},$$

where

$$T_{pq} = \sum_r \Delta^p u_r \Delta^q u_r;$$

$$r = t_0, t_0 + k_0 + 2, \dots, t + (j-1)(k+2) \leq n,$$

has the distribution function  $\alpha = P(L \leq L_0) = L_0^{(j'-1)}$  when the hypotheses made are true. Because the greatest possible value of  $j'$  is greater than that of  $j$ , when  $n$  is given, Johnson's test is more effective than Tintner's test. Johnson mentions also some new results about the value of the coefficient of correlation  $r(\Delta^p u_r, \Delta^q u_s)$  (for some combinations  $(p, q)$  and selections of sets  $(r, s)$ ), which could be utilised to get test criteria similar to  $L$ . L. Törnqvist.

Gartstein, B. N. On certain limit laws for the range. Doklady Akad. Nauk SSSR (N.S.) 60, 1119-1121 (1948). (Russian)

The following results are stated without proof. Let  $\xi_n$  and  $\eta_n$  be respectively the maximum and minimum of a sample of  $n$  numbers drawn from a continuous distribution. Let  $\varphi_n$  be the distribution function of the range  $\xi_n - \eta_n$ , and let  $\psi_n$  be the convolution of the distribution functions of  $\xi_n$  and  $-\eta_n$ . Then (1)  $\varphi_n(a_n u + b_n)$ , for properly chosen  $a_n > 0, b_n$ , converges to a limiting distribution function increasing at more than one point if and only if the same is true of  $\psi_n(a_n u + b_n)$ , with the same  $a_n, b_n$ , and in that case the limiting distribution functions are identical. Now suppose in addition to the hypothesis of the continuity of the original distribution that  $\xi_n$  and  $\eta_n$ , properly normalized and centered, have limiting distribution functions. These are known to be of one of three types, say  $\Phi_\alpha(u), \Psi_\alpha(u), \lambda(u)$ , where  $\alpha$  is a positive parameter [cf. Gnedenko, Ann. of Math. (2) 44, 423-453 (1943); these Rev. 5, 41]. Then (2) the possible limiting distribution types of  $\psi_n(a_n u + b_n)$  are  $\Phi_\alpha(u), \Psi_\alpha(u), \lambda(u), \Phi_\alpha(u) * \Phi_\alpha(au), \Psi_\alpha(u) * \Psi_\alpha(au), \lambda(u) * \lambda(au)$ , where  $a > 0$  is a constant. It follows that the only possible limiting distributions of the range are also these six types. This generalizes work of Gumbel [Ann. Math. Statistics 18, 384-412 (1947); these Rev. 9, 195]. J. L. Doob.

Finney, D. J. The Fisher-Yates test of significance in  $2 \times 2$  contingency tables. *Biometrika* 35, 145-156 (1948).

Seven pages of tables are given (with the method of calculation) for significance levels .05, .025, .01, .005, permitting the testing of  $2 \times 2$  contingency tables with no more than 15 observations in each cell of at least one margin.

C. P. Winsor (Baltimore, Md.).

Westenberg, J. Significance test for median and interquartile range in samples from continuous populations of any form. *Nederl. Akad. Wetensch., Proc.* 51, 252-261 (1948).

The author gives a nonparametric test for the difference between the medians of two equal-sized samples. If the number of members of the first sample smaller than the median of the combined sample is  $\frac{1}{2}N + \Delta$ , where  $N$  is the size of each sample, we have, on the null hypothesis,

$N!^2$

$(\frac{1}{2}N + \Delta)! (\frac{1}{2}N - \Delta)! (2N)!$

as the probability of the occurrence of  $\Delta$ . A short table is given of the values of  $\Delta$  for various sample sizes and significance levels. The same table can be applied to the interquartile range. C. P. Winsor (Baltimore, Md.).

\*Freeman, H. A., Friedman, Milton, Mosteller, Frederick, and Wallis, W. Allen, editors. *Sampling Inspection. Principles, Procedures, and Tables for Single, Double, and Sequential Sampling in Acceptance Inspection and Quality Control Based on Percent Defective*, by the Statistical Research Group, Columbia University. McGraw-Hill Book Company, Inc., New York and London, 1948. xx+395 pp. (3 plates). \$5.25.

This book presents, in a form suitable for the needs of industry, an extensive and highly useful set of sampling tables, together with an account of the construction of the tables and detailed instructions for their use in acceptance inspection or process control. The first 13 chapters [136 pages] contain an account of the principles of sampling inspection for attributes, and a discussion of questions connected with the choice of one among the more than 350 sampling plans for which tables are given in the book, and with the installation and operation of this plan. This part of the book is written for nonmathematical readers, and only very few formulae are given. In chapters 14-17 [78 pages], the construction of the sampling tables and the methods of their computation are discussed. Though these chapters will hardly be fully understood by a reader not familiar with the basic formulae of the subject, these are given only in chapter 17. The last part of the book [165 pages] contains tables and diagrams for a large collection of sampling plans.

H. Cramér (Stockholm).

Ghizzetti, Aldo. Sul problema del collaudo di partite di numerosi oggetti. *Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo* no. 164, 19 pp. (1945).

[Reprinted from *Atti della VII Riunione della Società Italiana di Statistica*, Roma, 1943.] The author discusses the problem of determining the quality of a large lot by means of sampling inspection. He is obviously unaware of the modern literature and modern methods. After rejecting the now generally abandoned "classical" procedure the author makes a remarkable, but rather crude, approach towards the modern concepts of errors and risks of the two kinds. No mathematical tools are developed.

W. Feller (Ithaca, N. Y.).

Kempthorne, Oscar, and Federer, Walter T. The general theory of prime-power lattice designs. I. Introduction and designs for  $p^n$  varieties in blocks of  $p$  plots. *Biometrics* 4, 54-79 (1948).

After an introductory discussion of  $p^n$  factorial designs in which the author gives formulae expressing the yield of a treatment-combination in terms of the interactions, the author proceeds to discuss quasifactorial designs. He observes that in these designs all interactions are of equal importance, since their significance is purely formal. Therefore the designs should be chosen so as to confound as far as possible different main effects and interactions in different replicates. The author then proposes systems of confounding which are nearly best in this sense. A detailed discussion of  $p^n$  designs is given for all  $n$  not exceeding 5 and the treatment for larger  $n$  is indicated. The author also discusses the analysis of the designs, utilizing intra- and inter-block information, and their relative efficiency.

H. B. Mann (Columbus, Ohio).

### Mathematical Economics

\*Massé, Pierre. *Les réserves et la régulation de l'avenir dans la vie économique. I. Avenir déterminé*. Actualités Sci. Ind., no. 1007. Hermann et Cie., Paris, 1946. v+149 pp.

This book is the first of a two part study concerning the optimum utilization of stocks of goods. Part I is based on the hypothesis of a determinate future (perfect foresight), while part II is based on the hypothesis of an uncertain (aleatory) future. The restricted framework of the analysis enables the author to identify the former with the short term, and the latter with the long term.

The determinist theory of regulation of reserves is first developed in terms of an indifference system through time. This analysis is achieved by a simple extension of the well-known Pareto-Walras theory of exchanges of distinct goods. The family of indifference functions is expressed by  $F(q_i) = \text{constant}$  ( $i = 1, 2, \dots, n$ ) representing successive time intervals. As usual the problem of the optimum consists in determining the maximum satisfaction subject to a budget condition. The  $n-1$  partial equilibrium relations and the budget equation determine the quantities  $q_i(t)$  representing the optimum distribution of a good in time.

For the major part of the study this general method is abandoned in favor of the narrower approach of profit maximization. The fundamental hypothesis becomes simply that to each function  $q(t)$  representing the flow of goods exchanged between two periods  $t_1$  and  $t_2$  there corresponds a measurable profit  $S$ . The arbitrage operation represented by  $q(t)$  may arise in a variety of forms: as pure speculation under which the stock  $Q$  is increased or decreased as prices change; as a simple time transformation of some natural flow  $[a(t)]$  such as annual harvests; or some time and physical transformation of one or more flows  $a_k(t)$ , such as a typical industrial operation.

The profit arising from an operation involving  $q(t)$  is defined as a functional dependent on all values of  $q(t)$  between  $t_1$  and  $t_2$ , and is written as  $S[q(t)]$ . It will suffice to take  $S = \int_{t_1}^{t_2} U(q, t) dt$ , where the form of  $U(q, t)$  depends on the demand function and the form of  $q(t)$ . For the case of simple exchange, for example,  $S = \int_{t_1}^{t_2} [pq - f(q)] dt - A$ , where

$f(g)$  represents the costs of exchange,  $A$  the fixed costs of storage, and  $p$  the selling price. The problem is to determine the function  $g(t)$  such that  $S[g(t)]$  is a maximum, subject to the budget restriction  $G[g(t)] = 0$ . The usual form of the budget equation is  $\int_0^t q dt - B = 0$ . A simple solution is given by the marginal analysis, with the maximum profit condition expressed by  $S'[g(t), \theta]/G'[g(t), \theta] = 0$ , where  $S'[g(t), \theta]$  is the derivative of the total profit function at the instant  $\theta$ . The author also presents a variant solution based on the calculus of variations which yields necessary conditions. The maximum profit solution is subject to additional constraining relations due to limitations on the capacity of accumulation of stocks and on the equipment. The form of these limitations and the solutions are considered in a series of applications. Finally, the question of a time discount on future profits is examined with obvious results.

M. P. Stoltz (Providence, R. I.).

\*Massé, Pierre. *Les réserves et la régulation de l'avenir dans la vie économique. II. Avenir aléatoire.* Actualités Sci. Ind., no. 1008. Hermann et Cie., Paris 1946. 230 pp.

The second part of this study of the optimum use of reserves is concerned with an uncertain future. Under the determinist hypothesis [see the preceding review] there exists a unique total profit for every value of the function  $g(t)$ . The simple objective of the regulation of reserves is to determine that value of  $g(t)$  which, under the given conditions, maximizes total profit. In the present case, to each operational rule there corresponds a probable value of the total profit, and the objective of the regulation is to maximize this mathematical expectation. Hence the solution is probabilistic. The solution given is subject to the following restrictions: (1) the uncertainties to which the variables are subject are limited to those which obey known (Gaussian) laws of probability; (2) the services rendered by the outflow ( $q$ ) are independent; (3) the time discount is ignored.

The problem may be put as follows. Consider the natural flow ( $x$ ) of a good into a reserve as a chance variable for which the probability law ( $K$ ) is known. There exists a stock ( $X$ ) at the beginning of a time period ( $t$ ), and a regulated flow out of the stock ( $q$ ). Take as known the function

( $U$ ) defining the total elementary profit attached to ( $q$ ) such that the function ( $S_t$ ) represents the profit attached to the residual stock at the end of ( $t$ ). The problem is considered in three parts: (1) the determination of a rule of regulation such that the probable value of successive profits is a maximum; (2) the determination of the laws of probability of the stocks and the flow ( $q$ ); (3) the evaluation of the effect of the regulation on the future profits.

The holder of a stock has, at the beginning of each period, to determine the rule of regulation defining the optimum instantaneous option for that elementary period. The unknowns of the problem are thus the outflow ( $q$ ) and the variations in the reserve  $\Delta X$  (or  $dX/dt$ ). The two equations necessary for their simultaneous solution are given by (1) an equation of conservation, e.g.,  $dX = (x - q)dt$ , and (2) the condition of marginal equilibrium, or a constrained relation if limitations on accumulation of stocks or equipment are present, which may be written in general form as  $s(X, x, t) = u(q, t)$ , where  $s$  is the partial derivative of the total expectation ( $S$ ) with respect to  $X$  and  $u$  is the partial derivative of the total profit function ( $U$ ) with respect to  $q$ . The expectation  $S$  is determined by the author by a process of iteration, reversed in time. The sum  $U_n + S_n$  represents the gain appropriate to the appearance of a value of  $x$  between  $x$  and  $dx$ , and  $(U_n + S_n)k_n dx$  represents the corresponding mathematical expectation. The total expectation  $S_{n-1} = E(U_n + S_n)$ , and since for this expression  $dS$  is alone a chance variable ( $x$  being no longer a chance value, but a datum) the equation reduces at the limit to  $Udt + E(dS) = 0$ .

The laws of probability of stocks and the flow ( $q$ ) may now be derived by the theory of chain probabilities, but the final result depends on the nature of the probability law assumed for the natural flow ( $x$ ). The author considers three cases: (1) successive independent flows, i.e.,  $k_n(x)$  is independent of the previous evolution of ( $x$ ); (2) discontinuous chain flows wherein the probability law depends on the value of  $x$  attained in the previous period; (3) continuous chain flows, which require the introduction of a stochastic differential equation. Only the first case is discussed fully, and that with respect to the linear case.

Finally, the author considers a solution from the point of view of the minimization of risk as an alternative to the maximization of an expectation.

M. P. Stoltz.

## TOPOLOGY

Errera, A. *Sur le problème des quatre couleurs. II.* Acad. Roy. Belgique. Bull. Cl. Sci. (5) 34, 65-84 (1948).

[For part I cf. same Bull. Cl. Sci. (5) 33, 807-821 (1947); these Rev. 9, 455.] Several formulas for the number of ways of coloring particular configurations are derived.

P. Franklin (Cambridge, Mass.).

Otter, Richard. *The number of trees.* Ann. of Math. (2) 49, 583-599 (1948).

Let  $A_n$  denote the number of nonhomeomorphic rooted trees having  $n$  nodes, including the root. [Cf. W. W. R. Ball, *Mathematical Recreations and Essays*, 11th ed., Macmillan, London, 1939; New York, 1947, p. 261; these Rev. 8, 440, where this is denoted by  $A_{n-1}$ .] In terms of  $S_n^{(1)} = A_{n+1-1} + A_{n+1-2} + A_{n+1-3} + \dots$  with the convention  $A_n = 0$  if  $n \leq 0$ , the author proves that

$$nA_{n+1} = 1A_1S_n^{(1)} + 2A_2S_n^{(2)} + \dots + nA_nS_n^{(n)}.$$

[The version of this formula at the bottom of p. 589 contains a misprint.] By considering the analytic behavior of

Cayley's function  $\sum A_{n+1}x^n = \prod (1-x^n)^{-1}$ , he deduces that, for large  $n$ ,  $A_n \sim 0.4399237 (2.95576)^n n^{-\frac{1}{2}}$ . For instance, this asymptotic expression gives 1708440 as the approximate value for  $A_{18} = 1721159$ .

He also considers the number of such trees where the degree of a node is not allowed to exceed a given integer  $m$ , while the degree of the root must not exceed  $m-1$ ; for instance, when  $m=4$  this is the number of structurally isomeric, mono-substituted, aliphatic hydrocarbons  $C_nH_{2n+1}X$ .

H. S. M. Coxeter (Toronto, Ont.).

Choquet, G. *Convergences.* Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.) 23, 57-112 (1948).

This paper is a point-set-theoretic study of correspondences between topological spaces, and consists of three parts. Part 1 deals with the notions involved in the following theorem, here presented as a representative example of many results. A relation  $R$  between two topological spaces  $X, Y$  is closed if and only if  $R$  is upper semi-continuous and

$R(x)$  is closed for each  $x$ . Here " $R$  is closed" means that the subset of  $X \times Y$  defining  $R$  is closed; and " $R$  is upper semi-continuous" means that for each  $x$ , and any  $y$  not in  $R(x)$ , there are neighborhoods  $V$  and  $W$  of  $x$  and  $y$  such that for  $x \neq z \in V$ ,  $R(z)$  does not meet  $W$ . The notions of upper limits and lower limits of a family of sets relative to a filter are defined and examined. A result not fitting into the familiar sequential precursors of these notions is that when the filter in question is ultra, the upper and lower limits of any family must coincide. Part 2 deals with two generalizations of  $T$ -spaces, suggested by the sometimes nontopological convergence spaces which arise when "continuous convergence" is defined in function spaces, and investigated to that end. Part 3 deals with the relation between abstract "contingents" and "paratingents" of Denjoy [Leçons sur le Calcul des Coefficients d'une Série Trigonométrique, v. 2, Gauthier-Villars, Paris, 1941; these Rev. 8, 260]. It is not feasible to define here the notions that enter into the many results obtained. Suffice it to say that the results deal with sets of the first and second categories.

R. Arens (Los Angeles, Calif.).

**Samuel, Pierre.** Ultrafilters and compactification of uniform spaces. *Trans. Amer. Math. Soc.* 64, 100-132 (1948).

The usual proof [Weil] that a uniform space  $X$  can be imbedded in a compact Hausdorff space involves the proof that  $X$  is completely regular, and thus the notion of real-valued continuous functions on  $X$ . The aim of this paper is to avoid the use of these functions in arriving at this and other results. After a preliminary discussion and abstraction of filters and ultrafilters [H. Cartan], the author shows how the set  $\Omega$  of ultrafilters in  $X$  becomes a compact Hausdorff space upon application of a modification of Stone's topology for the class of prime ideals in a Boolean ring. Using the uniform structure of  $X$ ,  $\Omega$  is partitioned into classes, giving a space  $X'$  in which  $X$  is naturally imbedded. The relation of these methods and results to those of Stone, Čech, Wallman and others is clearly set forth.

R. Arens (Los Angeles, Calif.).

**Sierpiński, W.** Sur un problème concernant les espaces métriques. *Mathematica, Timișoara* 23, 65-69 (1948).

All spaces are metric. A space  $E$  contains a model of a space  $M$  if there is a one-to-one mapping  $f$  of  $M$  into  $E$  such that  $r_E(f(p), f(q)) = kr_M(p, q)$  for all  $p, q \in M$ . If  $k=1$  and  $f(M)=E$ ,  $E$  and  $M$  are isometric. If  $\epsilon > 0$ ,  $f(M)=E$  and  $|r_E(f(p), f(q)) - r_M(p, q)| < \epsilon$  for all  $p, q \in M$ ,  $E$  and  $M$  differ by less than  $\epsilon$ . In connection with the problem, ascribed to Borsuk, of the existence of a semi-compact  $E$  ( $E = \sum_i E_i$ ,  $E$  compact) containing, for every compact  $M$ , an isometric  $E_M$ , the author proves the following results. (1) There is no compact space which contains a model of every denumerable space. (2) There is a compact space which contains a model of every finite space. (3) There is a semi-compact space which contains, for every finite space  $M$ , an isometric  $E_M$ . (4) For every space  $E$  and for every  $\epsilon > 0$  there is a space  $H$  with rational distances which differs from  $E$  by less than  $\epsilon$ . (5) There is a denumerable space  $H_0$  with rational distances which, for all denumerable spaces  $E$  and all  $\epsilon > 0$ , contains a space  $H_E$  differing from  $E$  by less than  $\epsilon$  [cf. P. Urysohn, *Bull. Sci. Math.* (2) 51, 43-64, 74-90 (1927)].

L. W. Cohen (Flushing, N. Y.).

**Kuratowski, Casimir.** Sur la topologie des espaces fonctionnels. *Ann. Soc. Polon. Math.* 20 (1947), 314-322 (1948).

Let  $X$  and  $Y$  be Fréchet  $L$ -spaces in which, if  $x_n$  does not converge to  $x$ , then some subsequence has no subsequence converging to  $x$  ( $L^*$ -spaces). Write  $f_n \rightarrow_B f$  ("continuous convergence") for  $f, f_n$  in  $C(X, Y)$ , the class of continuous  $Y$ -valued functions on  $X$ , if  $f_n(x_n) \rightarrow f(x)$  whenever  $x_n \rightarrow x$ . One of the author's results is that  $C(X, Y)$  is then an  $L^*$ -space. This fact was also shown by G. Birkhoff [Ann. of Math. (2) 35, 861-875 (1934)]. Other results are (a) in the equation  $g(x, t) = f(t)(x)$ , where  $g \in C(X \times T, Y)$ ,  $f \in C(T, C(X, Y))$ , the continuity of  $f$  is equivalent to that of  $g$ ; (b) when  $Y$  is metric,  $f_n \rightarrow_B f$  if and only if  $f_n(x) \rightarrow f(x)$  uniformly on sequentially compact sets. These results are related to known results [cf. R. H. Fox, *Bull. Amer. Math. Soc.* 51, 429-432 (1945); O. Frink, *Trans. Amer. Math. Soc.* 51, 569-582 (1942); R. Arens, *Ann. of Math.* (2) 47, 480-495 (1946); these Rev. 6, 278; 3, 313; 8, 165], but differ inasmuch as Kuratowski uses  $L^*$ -spaces rather than "topological" or " $T$ "-spaces. R. Arens (Los Angeles, Calif.).

**Borsuk, Karol.** On the topology of retracts. *Ann. of Math.* (2) 48, 1082-1094 (1947).

Every transitive class  $\mathfrak{R}$  of mappings generates a corresponding equivalence concept in the set of separable metrizable spaces. Space properties which are preserved by the mappings of a class  $\mathfrak{R}$  are called  $\mathfrak{R}$ -properties and form the subject matter of  $\mathfrak{R}$ -topology. These definitions lead to an instructive stratification of "topology" which is discussed at some length. The most exacting type of equivalence considered is topological or  $\mathfrak{H}$ -equivalence, which arises from the class  $\mathfrak{H}$  of topological mappings, and the least exacting is  $\mathfrak{C}$ -equivalence (two spaces are  $\mathfrak{C}$ -equivalent if each can be mapped continuously onto the other) which arises from the class  $\mathfrak{C}$  of continuous mappings.

The author is concerned with a previously unnoticed stratum which he calls the topology of retracts or the  $\mathfrak{R}$ -topology. It arises from the class  $\mathfrak{R}$  of continuous mappings which have right inverses. Two spaces are  $\mathfrak{R}$ -equivalent if each is homeomorphic to a retract of the other. The main theorem is the following: in order that every space  $\mathfrak{R}$ -equivalent to a (finite) polytope  $P$  be also homeomorphic to  $P$  it is necessary and sufficient that, for  $k \geq 2$ , no  $(k-1)$ -simplex of  $P$  be incident to exactly one of the  $k$ -simplexes of  $P$ . Auxiliary theorem: a (finite) polytope  $P$  is homeomorphic to one of its proper subsets if and only if, for some  $k \geq 1$ , there is a  $(k-1)$ -simplex of  $P$  incident to exactly one of the  $k$ -simplexes of  $P$ . It is noted that a space is homeomorphic to a retract of a polytope if and only if it is compact, locally contractible and finite dimensional, and to a retract of a compact convex set if and only if it is an absolute retract.

R. Fox (Princeton, N. J.).

**Borsuk, Karol.** An example of a simple arc in space whose projection in every plane has interior points. *Fund. Math.* 34, 272-277 (1947).

The results of this paper are contained in the following theorem. There exists in the  $n$ -dimensional Euclidean space  $C_n$  a simple arc  $B$  such that if  $B'$  is a subarc of  $B$  then there exists an  $n$ -simplex  $\Delta'$  such that every straight line in  $C_n$  which intersects  $\Delta'$  also intersects  $B'$ . Taking  $n=3$  then every projection (central or parallel) of  $B$  into a subset of a plane will contain interior points in that plane. The existence of such arcs indicates that the study of general knots

in 3-space by means of their projections is likely to be fruitless.

*J. H. Roberts* (Durham, N. C.).

**Bing, R. H. Some characterizations of arcs and simple closed curves.** Amer. J. Math. 70, 497-506 (1948).

The author obtains stronger characterizations of simple arcs and simple closed curves by using "cuts" in place of "separates." The set  $R$  separates  $A$  from  $B$  in the connected set  $M$  if  $M - R$  is the union of two mutually separated sets containing  $A$  and  $B$ , respectively. If  $M$  contains a continuum intersecting both  $A$  and  $B$  and  $R$  is a set containing neither  $A$  nor  $B$  but intersecting each continuum in  $M$  intersecting both  $A$  and  $B$ , we say that  $R$  cuts  $A$  from  $B$  in  $M$ . If a set separates (or cuts) two points from each other in  $M$ , it is said to separate (cut)  $M$ . Thus, among other things, it is shown that a nondegenerate compact continuum is a simple closed curve if it is cut by no single point but by each pair of points.

*A. D. Wallace.*

\***Aleksandrov, P. S. Kombinatornaya Topologiya.** [Combinatorial Topology]. OGIZ, Moscow-Leningrad, 1947. 660 pp.

This book is designed as an introduction to that field of topology which can be treated by the methods of homology theory. It is addressed to every student of mathematics who has even a small acquaintance with set theory and the elements of Abelian group theory. In the first third of the book, preliminary to a formal homology theory, even this little is not a specific prerequisite; all concepts introduced are carefully defined and clarified by many remarks and examples. The examples, of themselves, furnish a fine background of experience in topology.

It is not until chapter 8 [page 292] that Betti-groups are defined and the machinery of homology (and cohomology) theory set in motion. The first six chapters, comprising parts I and II, are devoted to a central core of topological ideas. These are, briefly, the Jordan curve theorem [chapter 2], the two-dimensional manifolds [chapter 3], the Sperner lemma and Stone-Mason theorem with a number of topological consequences [chapter 5], and dimension theory [chapter 6]. Chapters 1 and 4 are generally introductory, chapter 1 of a great variety of topics relating to topological spaces and continuous mappings; the other, of topics relating to geometric complexes and their realization in Euclidean space. Were these two chapters shorn of material not needed until much later in the book, parts I and II would constitute of themselves a most excellent and elementary introduction to general topology, carefully expounded and thoroughly illuminated by remarks, examples and pictures. The proofs are thorough, and the techniques employed are such as to predispose the reader to the later combinatorial generalizations.

The central chapters, constituting part III, develop homology and cohomology theory for locally finite abstract complexes, and for compacta. The coefficient groups are arbitrary Abelian groups or fields, regarded as not topologized, and the theory is restricted to finite chains. Much attention is given to polyhedra and operations on polyhedra, in keeping with the general emphasis throughout the text upon geometric content. A number of illustrative examples of the calculation of homology groups are shown using the important coefficient groups: integers, integers mod  $m$ , rationals, and reals mod 1. The homology theory of compacta is based upon the notion of proper-cycles, akin to the (unmentioned) cycles of Vietoris. It is an instance of the

deliberate style of the book that the invariance of the homology groups of polyhedra is proved directly, using subdivisions and simplicial mappings, and is also later deduced from the invariance of the homology groups of compacta. The selection of canonical bases for homology cycles, and the relations between homology groups of the same complex when taken over different coefficient groups, are very thoroughly studied.

The culmination of the book is the Alexander-Pontrjagin duality theorem, to which much of part IV is devoted. This theorem is stated as an isomorphism between the (respective) cohomology groups of a compactum in spherical  $n$ -space and the associated homology groups of its complement. Both groups are taken over the same (discrete) coefficients. In this way the author is able to avoid the relatively complicated theory of topological groups and their character groups. The original Alexander duality theorem, restricted to topological complexes and Betti numbers, is proved independently of the general theorem. Many of the preliminaries to a proof of the general duality theorem are studied against the background of the "homological manifolds" characterized by their local homology groups, and it is here that the Poincaré-Veblen duality for manifolds is proved.

The final chapters [part V] are devoted to an application of homology theory to the mappings of polyhedra and to fixed point theorems. These chapters the author speaks of as a revision and translation, respectively, of corresponding chapters in the Topologie of Alexandroff and Hopf [Springer, Berlin, 1935] and credits in the original to his collaborator. Two appendices, on the theory of Abelian groups and on  $n$ -dimensional spaces, are substantially the same as in the older book.

The author is at pains to disclaim completeness in the variety of topics covered, or the development of any topic, or in historical reference. The book is presented at every point as an introduction to its subject matter. Even the brief indications of content given above will suggest, however, that it is a very substantial text. More might have been crowded into the same number of pages, but not profitably for the younger students for whom it is intended.

*L. Zippin* (Flushing, N. Y.).

**Dyson, F. J. A theorem in algebraic topology.** Ann. of Math. (2) 49, 75-81 (1948).

For use in a problem in the theory of numbers, the author proves a combinatorial duality theorem. Before stating the theorem, it is necessary to give some definitions. Let  $T_1$  be a locally finite simple complex. Let  $S$  be a projective space. To each point  $\alpha$  of  $S$ , let there correspond a closed subcomplex  $R_\alpha$  of  $T_1$ , the union of the family  $\{R_\alpha\}$  being all of  $T_1$ . For a  $k$ -spread  $\pi_k$  of  $S$ , define  $C(\pi_k)$  to be the subcomplex of  $T_1$  consisting of all those  $R_\alpha$  for which  $\alpha$  does not lie in  $\pi_k$ . For each integer  $l$ , define  $T_l$  to be the subcomplex of  $T_1$  consisting of each element of  $T_1$  which lies in at least  $l$  different subcomplexes  $R_\alpha$  for which the subscripts are linearly independent in  $S$ . Finally denote by  $C_\alpha(\pi_k)$  and  $T_{l\alpha}$  the corresponding augmented complexes. Now the statement of the theorem is: if, for a given integer  $m \geq 0$ ,  $C_\alpha(\pi_k)$  is acyclic up to dimension  $(m-k-1)$  for  $k = -1, 0, 1, \dots, m$  for every  $k$ -spread  $\pi_k$  in  $S$ , then  $T_{l\alpha}$  is acyclic up to dimension  $(m-l-1)$  for  $l = 1, 2, \dots, m+2$ . In particular, there are  $m+2$  members of  $\{R_\alpha\}$ , with linearly independent subscripts, which have a nonvacuous common part.

*E. G. Begle* (New Haven, Conn.).

**Pontryagin, L. S. Topological duality theorems.** *Uspehi Matem. Nauk* (N.S.) 2, no. 2(18), 21–44 (1947). (Russian)

This is a fairly detailed article on the substance of a paper of the author [Math. Ann. 105, 165–205 (1931)]. The older paper is substantially recast, the proofs being "strongly modernized" (in the author's words), to make use of cohomology theory and the theory of character groups. Both of these theories were developed subsequently to the original paper. The reader is expected to be familiar with the material in the first four chapters of Seifert and Threlfall's "Lehrbuch der Topologie" [Teubner, Leipzig-Berlin, 1934] and the first chapter of the author's paper on topological groups [Ann. of Math. (2) 35, 361–388 (1934)]. The needed combinatorial concepts are reviewed in the paper but none pertaining to character group theory are developed, the reader being referred to the paper cited for each needed theorem.

*L. Zippin* (Flushing, N. Y.).

**Bockstein, M. Sur la dimension module  $m$ .** *Fund. Math.* 34, 306–310 (1947).

Let  $A$  be a closed set in Euclidean  $n$ -space,  $m \geq 2$  an integer. Let  $\Delta_m A$  be the dimension of  $A$  modulo  $m$  in the sense of P. Alexandroff [Math. Ann. 106, 161–238 (1932)]. The author proves that  $\Delta_m A = \max \Delta_k A$  for  $k$  a prime divisor of  $m$ .

*J. H. Roberts* (Durham, N. C.).

**Bockstein, M. Sur la dimension par rapport à la dominante.** *Fund. Math.* 34, 311–315 (1947).

The author introduces the following definition closely related to the notions of power cycles and dimensional dominants introduced by Alexandroff [reference in the preceding review]. The dimension of  $A$  with respect to the dominant  $m$ ,  $\dim_m A$  ( $m \geq 2$  being an integer) is the greatest integer  $q$  with the following property: there exists a closed subset  $B$  of  $A$  for which there exists an  $\epsilon > 0$  such that for each  $\delta > 0$  there is a positive integer  $k$  such that there is on  $A$  a  $q$ -dimensional  $(\bmod m^k)$   $\delta$ -cycle relative to  $B$  which is not  $\epsilon$ -homologous to zero, relative to  $B$ . The main results are that the ordinary dimension of  $A$  is the maximum  $\dim_m A$  for  $m \geq 2$ , and that  $\dim_m A = \max \dim_p A$  for  $p$  a prime divisor of  $m$ .

*J. H. Roberts* (Durham, N. C.).

**Reeb, Georges. Remarque sur les variétés feuilletées contenant une feuille compacte à groupe de Poincaré fini.**

*C. R. Acad. Sci. Paris* 226, 1337–1339 (1948).

In a previous note [same *C. R.* 224, 1613–1614 (1947); these Rev. 8, 595] the author introduced the notion of a leaved manifold (variété feuilletée) and proved a theorem about  $(n-1)$ -leaved manifolds. He now shows that the corresponding theorem for  $q$ -leaved manifolds,  $q < n-1$ , does not hold, by constructing a closed  $(n-2)$ -leaved manifold for which some, but not all, leaves are compact with finite fundamental group.

*H. Samelson*.

**Ehresmann, Charles. Sur les variétés plongées dans une variété différentiable.** *C. R. Acad. Sci. Paris* 226, 1879–1880 (1948).

A deformation theorem on the mappings of one fibre space into another is stated. Applications are given to fibre spaces formed from  $p$ -dimensional elements in an  $n$ -dimensional differentiable manifold and to related spaces.

*H. Whitney* (Cambridge, Mass.).

**Youngs, J. W. T. The extension of a homeomorphism defined on the boundary of a 2-manifold.** *Bull. Amer. Math. Soc.* 54, 805–808 (1948).

If  $M$  and  $M'$  represent homeomorphic orientable 2-manifolds, the author proves that a homeomorphism of the boundary of  $M$  into  $M'$  can be extended to a homeomorphism of  $M$  into  $M'$  if, and only if, a concordant orientation is carried into a concordant orientation. A concordant orientation on an orientable 2-manifold  $M$  is defined to be an orientation on each Jordan curve  $J_i$  ( $i=1, \dots, n$ ) of the boundary of  $M$  such that the orientation induced on  $M$  by the orientation on any  $J_i$  is independent of the value  $i$ . If  $M$  and  $M'$  are nonorientable 2-manifolds, it is shown that the homeomorphism can always be extended.

*V. W. Adkisson* (Fayetteville, Ark.).

**Newman, M. H. A. Boundaries of ULC sets in Euclidean  $n$ -space.** *Proc. Nat. Acad. Sci. U. S. A.* 34, 193–196 (1948).

An example is constructed of an  $(n-1)$ -manifold with nonvanishing fundamental group  $\pi_1$ , imbedded rectilinearly in the Euclidean  $n$ -space  $E^n$  and forming the boundary in  $E^n$  of a domain whose homotopy groups (including  $\pi_1$ ) are all zero. This leads to an example of an open subset of  $E^n$  that is ULC\* but whose boundary is not 1-LC (LC denoting local connectedness in the sense of homotopy), thus settling negatively a question raised by S. Eilenberg and the reviewer [Amer. J. Math. 64, 613–622 (1942), problem 2; these Rev. 4, 87].

*R. L. Wilder*.

**Supnick, Fred. On the perspective deformation of polyhedra.** *Ann. of Math.* (2) 49, 714–730 (1948).

In Euclidean 3-space, let  $P$  be a polyhedron homeomorphic to a 2-sphere, having triangular faces, and star-shaped relative to an interior point  $O$ . Is it possible to deform  $P$  into a convex polyhedron by motions of the vertices along rays from  $O$ , all faces remaining rectilinear triangles throughout the deformation? The author reduces this problem to the case of a simple concave polyhedron; that is, one differing from its convex hull by a single solid tetrahedron. Despite this significant reduction, the question remains open for the general simply convex octahedral case (six vertices each incident with four edges) and for simply concave polyhedra in general of more than six vertices. For polyhedra of fewer than seven vertices, save in the octahedral case, an affirmative answer is established.

*S. S. Cairns* (Urbana, Ill.).

**Moise, Edwin E. An indecomposable plane continuum which is homeomorphic to each of its nondegenerate subcontinua.** *Trans. Amer. Math. Soc.* 63, 581–594 (1948).

The author answers a long-standing question due to S. Mazurkiewicz [Fund. Math. 2, 286 (1921)] by constructing a continuum having the properties stated in the title. The continuum, called a pseudo arc, is the common part of an infinite sequence of chains of closed regions. One is reminded of Knaster's "method of bands," and the author's example is very similar, if not in fact actually homeomorphic (unsolved problem), to a continuum described by Knaster [Fund. Math. 3, 247–286 (1922)].

*J. H. Roberts* (Durham, N. C.).

## GEOMETRY

Prenowitz, Walter. *Total lattices of convex sets and of linear spaces.* Ann. of Math. (2) 49, 659-688 (1948).

The object of this paper is to give a characterization of linear geometries as lattices. It is a sequel to the author's earlier papers [Amer. J. Math. 65, 235-256 (1943); Trans. Amer. Math. Soc. 59, 333-380 (1946); these Rev. 4, 251; 7, 375]. A linear geometry is defined in terms of an order relation  $(abc)$ , which intuitively may be taken to mean that  $a$ ,  $b$  and  $c$  are collinear and  $b$  is between  $a$  and  $c$ . We say that  $a$  is "perspective" to  $b$ . This order relation is subject to five postulates: (O1)  $(abc)$  implies that  $a$ ,  $b$  and  $c$  are distinct; (O2)  $(abc)$  implies  $(cba)$ . If  $a \neq b$  the set of all  $x$  such that  $(xab)$ ,  $x=a$ ,  $(axb)$ ,  $x=b$ , or  $(abx)$  is called the line  $ab$ . (O3) There is a unique line containing  $c \neq d$ . (O4') The point  $b$  can be reached from  $a$  in a finite number of perspective steps. In other words, any two points are projective. (O6': Transversal property) If  $(yxc)$  and  $(ayb)$ , then there exists a point  $z$  such that  $(axz)$  and  $(bzc)$ .

A lattice can be defined on a linear geometry by defining the union  $a \cup b$  of two points to be the set of all  $x$  such that  $x=a$ ,  $(axb)$  or  $x=b$ . This lattice is complete and satisfies the following five properties. (I: Point existence)  $a < b$  implies that there exists a point  $p$  such that  $a < a \cup p \leq b$ . (II: Finiteness of dependence) Let  $\alpha$  be a set of points and  $p \leq \sup \alpha$ . Then  $p \leq p_1 \cup p_2 \cup \dots \cup p_n$  for some  $n$  points  $p_i$  of  $\alpha$ . The notion of linear dependence is introduced and a "closed" element defined as one containing with each set those elements linearly dependent on it. (III: Transitivity of dependence) If  $p$  is dependent on  $q_1, q_2, \dots, q_m$  and if  $q_i$  is dependent on  $r_1, r_2, \dots, r_n$  for each  $i$ , then  $p$  is dependent on  $r_1, r_2, \dots, r_n$ . (IV: Quasimodularity) If  $c$  is closed and  $a \leq c$  then  $(a \cup b) \cap c = a \cup (b \cap c)$ . (V: Quasimodularity) There are no complete congruence relations (except the two trivial ones). Furthermore, these five properties with completeness characterize this lattice.

If in addition the lattice is modular, then the geometry is projective. In this case the properties can be restated: the linear spaces of a projective geometry form a modular exchange lattice which is quasi-simple.

H. H. Campaigne (Arlington, Va.).

Tietze, Heinrich. *Zur Analyse der Lineal- und Zirkel-Konstruktionen. I.* S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. 1944, 209-231 (1944).

The author remarks that the usual theory of constructions with straight-edge and compasses (or other instruments) tacitly assumes the use of an instrument which decides the order of three points on a straight line. If such a decision is not permitted (which implies that the two points of intersection of a line and a circle or of two circles cannot be distinguished), not all classical constructions can be performed. The following criterion is given without proof. In order that a point  $(x, y)$ , a line  $ux+vy+w=0$  or a distance  $r$  can be constructed from  $n+1$  given points  $(0, 0)$ ,  $(a_1, b_1)$ ,  $\dots$ ,  $(a_n, b_n)$  by means of straight-edge and compasses only (i.e., without any order decisions), it is necessary and sufficient that  $x = \sum_i \rho_i a_i$ ,  $y = \sum_i \rho_i b_i$ ;  $u:v:w = \sum_i \rho_i a_i : \sum_i \rho_i b_i : \rho_0 p_{11}$ ;  $r^2 = \rho_0 p_{11}$ , where  $\rho, \rho_0, \dots, \rho_n$  are in the field generated by the ratios  $\rho_{\mu\nu}/p_{11}$  ( $\rho_{\mu\nu} = a_\mu a_\nu + b_\mu b_\nu$ ). Example: if two points  $(0, 0)$ ,  $(1, 0)$  are given, only those distances can be constructed which are square roots of rational numbers; thus  $\sqrt{3}$  can be constructed,  $1 + \sqrt{3}$  cannot.

F. A. Behrend (Melbourne).

Rossier, Paul. *Esquisse d'une théorie de l'équerre.* Elemente der Math. 3, 49-51 (1948).

The author discusses the constructions based on the following operations with an angle. (1) (a) Apply one side of the angle to a given straight line. (b) Let one side of the angle slide along a straight edge so as to make the second side pass through a given point. (2) Let one side of the angle slide along a given straight line so as to make a fixed point of the side other than the vertex coincide with a given point of the line. (3) Move the angle so that its sides pass through two given points and the vertex falls on a given line. Operation (1) allows those constructions which correspond analytically to the rational operations; (1) and (2) are equivalent to the use of straight edge and graduated scale; (1), (2) and (3) give the ruler and compass constructions. If the use of  $n-1$  angles is permitted algebraic equations of degree  $n$  can be solved. F. A. Behrend.

Bartolo, M. *Il problema di Delo ed una sua applicazione acustica.* Boll. Un. Mat. Ital. (3) 2, 235-238 (1947).

The duplication problem is solved approximately by constructing a segment  $x$  such that  $x:0.9 = 0.7:0.5$ . As  $x=1.26$  and  $2^1=1.259921 \dots$ , the error is less than  $10^{-4}$ . The acoustical "application" consists in a construction of the distances of the frets for the fingerboard of a string instrument so as to obtain the tempered semitone scale (two successive frequencies of which are in the ratio  $1:2^{1/12}$ ).

F. A. Behrend (Melbourne).

Giršovič, M. V. *On geometrical constructions in the Lobachevsky plane.* Doklady Akad. Nauk SSSR (N.S.) 60, 757-759 (1948). (Russian)

The author proves, by geometric arguments only, the following two theorems. (1) Geometrical constructions with Lobachevsky ruler and compasses, in the sense of Poincaré, are geometrical constructions with Euclidean ruler and compasses. (2) Geometrical constructions in the Lobachevsky plane in the sense of Poincaré, performed with Euclidean ruler and compasses, can be performed with Lobachevsky ruler and compasses. The first theorem gives no difficulties. The second is divided in two parts. (a) The geometrical constructions can be performed with Lobachevsky ruler, compasses and instruments drawing hypercycles and horocycles. (b) The geometrical constructions in the Lobachevsky plane with the four instruments mentioned above can be performed with Lobachevsky ruler and compasses.

H. A. Lauwerier (Amsterdam).

Decuyper, Marcel. *Sur l'hypocycloïde à trois rebroussements.* Ann. Sci. École Norm. Sup. (3) 64 (1947), 227-246 (1948).

The author gives an elementary treatment of the hypocycloid  $H_3$  with three cusps, based upon the following definition. If two points move on a circle, the ratio of their velocities being  $-1$ , then the line connecting them envelopes an  $H_3$ . The following properties are deduced. The segments cut out on a fixed tangent by any pair of perpendicular tangents are concentric [Hadamard]. The segments on the tangents cut out by the curve have the same length [Cremona]. The locus of the points of intersection of two perpendicular tangents is the tritangent circle. The author indicates the well-known kinematic generation as well as a new tangential generation of the  $H_3$ . Finally he investi-

gates the family of  $H_3$ 's inscribed in a given triangle, furthermore, the  $H_3$  inscribed in a quadrangle or tangent to the sides and altitudes of a triangle. His method is quite elementary; not even coordinate geometry is used. [The reviewer remarks that most of the elementary properties of the  $H_3$  become evident if both kinematic generations are used simultaneously.]

E. Egerváry (Budapest).

**Ladopoulos, Takis.** Some theorems on cyclic polygons inscribed in a circle. Amer. Math. Monthly 55, 301-307 (1948).

In this paper, a "cyclic  $n$ -gon" is projectively equivalent to a regular polygon. For the case that the circumscribing conic is a circle, the author establishes the existence of generalizations of the Lemoine (symmedian) point and the Lemoine line, the Brocard points, angle and circle, the Lemoine circles, and even the Tucker circles.

For  $n=4$ , the results for the so-called harmonic quadrangle have been on record since 1885 [cf. Johnson, Modern Geometry, Houghton Mifflin, Boston, 1929, footnote p. 301]. For the general case the results are probably new.

R. A. Johnson (Brooklyn, N. Y.).

**\*Blaschke, Wilhelm.** Projektive Geometrie. Wolfenbütteler Verlagsanstalt, Wolfenbüttel-Hannover, 1947. 160 pp.

This is a very readable monograph on analytic projective geometry, with 61 figures and a good index, but no exercises. Chapter I contains an interesting account of the history and bibliography of the subject, including such names as Dürer and Leonardo da Vinci. In chapter II homogeneous coordinates are derived in the classical manner from Cartesian coordinates, and there is a neat treatment of matrices, linear equations, collineations and correlations. Chapter III, dealing with cross ratio, contains a rigorous proof that the group of collineations is derived from the group of linear transformations by adjoining the automorphisms of the coordinate field. In chapter IV, conics and polarities are treated together, the latter being expressed in the form  $\sum a_{\mu\nu}x_\mu x_\nu = 0$ . An analytic proof is given that the Desargues configuration is self-polar. The connection between Pappus' theorem and commutativity is well described. Pascal's theorem is proved in four different ways, two synthetic and two analytic. This leads naturally to a discussion of Dandelin's skew hexagon of generators of a quadric. We are also introduced to Cayley's projective metric, with the Euclidean metric as a limiting case. This is followed by some metrical properties of conics, such as Poncelet's derivation of foci from the circular points, and Ivory's elegant theorem to the effect that the curvilinear rectangle formed by four confocal conics has equal straight diagonals. The fifth and sixth chapters deal with line geometry and quadrics, respectively. The latter includes a brief description of the Veronese surface in projective 5-space and of its 9-dimensional analogue. Chapter VII is an analytic treatment of non-Euclidean geometry, using complex numbers and quaternions. This includes an excellent account of Study's representation of lines in elliptic space by point pairs on two spheres. The octahedral group of rotations is related to Stephanos' configuration of desmic tetrahedra, and thence to the regular 24-cell in Euclidean 4-space [cf. Coxeter, "Regular Polytopes," London, 1948, p. 149]. Finally, in chapter VIII, all the techniques developed in the earlier part of the book are brought together for the discussion of Möbius' configuration of eight points

and eight planes in real projective 3-space. The theorem that seven of the incidences imply the eighth is proved in five different ways.

Two mistakes were noticed. On page 56 the self-duality of the Pappus configuration is denied; and on page 151 Kummer's configuration is said to contain fifteen (instead of thirty) Möbius configurations.

The book is enlivened by a wealth of historical information, such as the remark that determinants were discovered not only by Leibniz but also by his Japanese contemporary Seki Shinsuke Kowa. However, it is strange to find Desargues' involution theorem ascribed to Sturm [p. 67].

H. S. M. Coxeter (Toronto, Ont.).

**Urban, Alois.** The locus of the centers of similar conics in a net of conics. Časopis Pěst. Mat. Fys. 72, D67-D74 (1947). (Czech)

The author gives proofs of well-known theorems for the centers of similar conics in the net of conics which have the same polar triangle. He uses the quadratic transformation in which the center of a conic in the net corresponds to the center of a point-involution on a circumference. This involution is the projection (from a point of the circumference) of the involution of polar points induced on the line at infinity by the conic considered.

F. Výchichlo.

**Ionescu, D. V.** Une application d'une équation fonctionnelle. Acad. Roum. Bull. Sect. Sci. 24, 573-576 (1943).

Let  $C$  be a plane curve referred to  $\lambda$  as parameter. Let, for each  $h$ , the area between  $C$  and the chord joining the points with parameters  $\lambda-h$  and  $\lambda+h$  be independent of  $\lambda$ . Then  $C$  is a conic.

F. John (New York, N. Y.).

**Rosina, B. A.** Sopra un modo semplice e rapido per scrivere l'equazione canonica di una conica. Ann. Univ. Ferrara 5, 153-158 (1947).

**Yang, Chung-Tao.** Projective collineations in a finite projective plane. Acad. Sinica Science Record 2, 157-164 (1948).

If  $P$  is the projective plane with coordinates from the Galois field of order  $p^n$ , then one may classify its collineations according to the structure of their systems of fixed elements. The author computes the number of collineations in each of these classes.

R. Baer (Urbana, Ill.).

**Kuo, P. T.** Projective correspondences in the finite projective geometry  $PG(3, 2)$ . Acad. Sinica Science Record 2, 171-178 (1948).

This is an investigation of the possible types of projectivities of the three-dimensional space with coordinates from the two-element field. The treatment would have been simpler if the author had not avoided the use of coordinates.

R. Baer (Urbana, Ill.).

**Rodéja F., E. G.-** Generalization of Pascal's theorem to hyperspaces. Revista Mat. Hisp.-Amer. (4) 8, 23-45 (1948). (Spanish)

Dans la généralisation dont il est question l'auteur substitue à la conique de l'énoncé usuel une courbe rationnelle normale  $C_n$  [ $x_0=1, x_1=\lambda, x_2=\lambda^2, \dots, x_n=\lambda^n$ ] de l'espace  $E_n$ , et aux six sommets de l'hexagone inscrit  $n+1$  couples de points  $(P_1, \dots, P_n, P_{n+1}; P_{n+2}, \dots, P_{2n+2})$ , les couples successifs étant  $(P_1, P_{n+2}), (P_2, P_{n+3}), \dots$ . La notion de système de trois points alignés est remplacée par celle de système de  $n+1$  espaces linéaires  $E_{n-2}$  associés, un tel

système étant caractérisé par la propriété que toute droite de  $E_n$  qui coupe  $n$  des espaces linéaires qui le constituent coupe aussi le  $(n+1)$ ième.

Si l'on considère  $n$  points successifs quelconques dans la permutation circulaire des indices des  $2n+2$  points  $P$  pris sur  $C_n$  et les  $n$  points correspondants dans les  $n+1$  couples précédemment indiqués, chacun de ces deux groupes de  $n$  points définit un hyperplan, et l'intersection de ces deux hyperplans est un  $E_{n-2}$ . En variant la permutation on obtient ainsi  $n+1$  espaces  $E_{n-2}$ , et le théorème généralisant celui de Pascal qui fait l'objet du mémoire consiste en ce que ces  $n+1$  espaces  $E_{n-2}$  sont associés au sens indiqué. La démonstration est basée sur deux lemmes de natures respectives algébrique et combinatoire intéressants par eux-mêmes.

La notion de système de  $n+1$  droites associées, duale de celle de système de  $n+1$  espaces  $E_{n-2}$  associés, conduit aussitôt au théorème dual du précédent. Soient donnés, pour une courbe rationnelle normale,  $n+1$  paires d'hyperplans osculateurs  $(\pi_1, \dots, \pi_{n+1}; \pi_{n+2}, \dots, \pi_{2n+2})$  acoupés comme le sont plus haut les points  $P$ ; le point commun à  $n$  hyperplans successifs dans la permutation des indices et le point commun aux  $n$  hyperplans complétant les couples auxquels les précédents appartiennent déterminent une droite; en variant la permutation des indices on obtient  $n+1$  droites, et ces  $n+1$  droites sont associées.

P. Vincensini (Besançon).

**Vančura, Zdeněk.** Conics in the hyperbolic non-Euclidean plane. *Acta Fac. Nat. Univ. Carol.*, Prague no. 182, 37 pp. (1948). (Czech. French summary)

Conics are defined by embedding the hyperbolic plane into the real projective plane and considering those conics of the projective plane which are wholly or partially interior to the absolute. Such a conic determines a polarity whose product with the absolute polarity is a certain collineation. A first classification is obtained by considering the nature of this collineation as given by its Segre characteristic:  $[111]$  or  $[21]$  or  $[(11)1]$  or  $[3]$  or  $[(21)]$ . Thus  $[(21)]$  is the horocycle, which superosculates the absolute and has just one branch, like a parabola. The osculating conic  $[3]$  again has a single branch, but this behaves like a parabola at one end and like a hyperbola at the other, having an asymptote there. The remaining three types undergo further subdivision according to the number of real intersections with the absolute. Thus  $[(11)1]$  is either a circle or a hypercycle (equidistant curve), having double contact with the absolute. The symbol  $[21]$ , denoting simple contact, covers three types: one resembling a parabola, one having one branch with two asymptotes, and one having two branches with one asymptote for each. Finally,  $[111]$  covers four types. One resembles the second kind of  $[21]$ , and one is a simple oval like the ordinary ellipse, while the remaining two both have four real, distinct intersections with the absolute, and consequently both have two branches and four asymptotes. The four points at infinity form a quadrangle whose diagonal triangle has one proper vertex and two proper sides, naturally called the center and axes of the conic. The conic is symmetrical about its axes in the ordinary way, and lines through the center are called diameters. The ultimate distinction is made by examining the involution of conjugate diameters, which may be elliptic or hyperbolic. Thus altogether the hyperbolic plane admits eleven kinds of conic, in the sense in which the elliptic plane admits two (the circle and the general conic), the affine

plane three (the ellipse, parabola and hyperbola), and the Euclidean plane four. H. S. M. Coxeter (Toronto, Ont.).

**\*Seidel, Johan Jacob.** De Congruentie-Orde van het Elliptische Vlak. [The Congruence Order of the Elliptic Plane]. Thesis, University of Leiden, 1948. iv+71 pp. (Dutch. English summary)

This thesis contains the proofs promised in a recent note [J. Haantjes and J. Seidel, Nederl. Akad. Wetensch., Proc. 50, 892-894 = Indagationes Math. 9, 403-405 (1947); these Rev. 9, 299] that the congruence order of the elliptic plane, with respect to the class of metric spaces, is seven (that is, seven is the smallest number with the property that an arbitrary metric space is congruent with a subset of the elliptic plane whenever each seven of its points are). Though the algebraic-geometric methods by which this result is established occasionally lead to proofs involving the examination of many cases and subcases (the longest proof covers more than seven pages) and they could hardly be carried through in elliptic three-space, the arguments (elementary in character) are presented in a clear and orderly manner. The reader must be referred to this painstaking work itself for details. L. M. Blumenthal (Columbia, Mo.).

**Fog, David.** Centroids and medians in spherical space.

I. *Mat. Tidsskr. B.* 1947, 41-47 (1947). (Danish)

**Fabricius-Bjerre, Fr.** Centroids and medians in spherical space. II. *Mat. Tidsskr. B.* 1947, 48-52 (1947). (Danish)

The centroid of a finite point set in a spherical space may be defined inductively in the following manner. For two (not opposite) points it is the midpoint of the segment joining them. For a set consisting of  $n$  points it is the common point of the  $n$  segments joining one of the points with the centroid of the remaining  $n-1$ . A median of a finite point set is a segment joining the centroids of two complementary subsets. Both authors prove a number of properties and formulae concerning centroids and medians analogous to (mostly known) properties of the corresponding Euclidean concepts, Fog by means of spherical trigonometry and the theorems of Menelaos and Ceva in spherical geometry and Fabricius-Bjerre by considering the spherical space as a hypersurface in a Euclidean space and by central projection of suitable Euclidean configurations.

W. Fenchel (Copenhagen).

### Convex Domains, Extremal Problems

**Hadwiger, H.** Über die Zerstücklung eines Eikörpers. *Math. Z.* 51, 161-165 (1948).

K. Borsuk [Fund. Math. 20, 177-190 (1933)] conjectured that it is possible to decompose every point set with diameter 1 in  $n$ -dimensional Euclidean space into  $n+1$  subsets each of which has a diameter smaller than 1. The author proves this for convex bodies with a boundary sufficiently regular so that a sphere can roll freely in the interior. Let  $r$  be the radius of the largest sphere with this property, i.e., the minimum of the principal radii of curvature of the boundary. Then it is shown that the decomposition may be chosen so that the diameters of the  $n+1$  subsets are smaller than  $1-2r[1-(1-n^2)^{1/2}]$ . W. Fenchel.

Fejes Tóth, László. *On the densest packing of convex domains.* Nederl. Akad. Wetensch., Proc. 51, 544-547 = Indagationes Math. 10, 188-192 (1948).

Let  $C$  be a plane convex domain symmetric with respect to the origin. The plane is covered by nonoverlapping domains each of which is a displacement of  $C$  whose centers form a lattice. Let  $d$  denote the ratio of the area so covered to the total area of the plane. The paper gives an elementary proof that for a given  $C$  the above lattice can always be chosen so that  $d > \frac{1}{2}\sqrt{3}$ . *D. Derry.*

Kameneckii, I. M. *Solution of a geometrical problem of L. Lyusternik.* Uspehi Matem. Nauk (N.S.) 2, no. 2(18), 199-202 (1947). (Russian)

This note shows that if a given triangle has an angle incommensurable with  $\pi$ , then the inscribed circle is the only Jordan curve about which the triangle can be rotated so that it is always circumscribed about the curve. Fujiwara [Sci. Rep. Tôhoku Imp. Univ., Ser. 1, 4, 43-55 (1915), Satz II, p. 50] proves this theorem for polygons of  $n$  sides where  $n$  is an arbitrary integer greater than 2 but not 4.

*M. M. Day* (Princeton, N. J.).

Inzinger, R. *Sui diametri coniugati delle ovali a centro.* Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 3, 293-295 (1947).

Two diameters of a plane oval with a center are called conjugate with respect to the oval if the tangents at the end points of one are parallel to the other. Using previous results [Akad. Wiss. Wien, S.-B. IIa, 147, 235-242 (1938); 155, 1-14 (1946); these Rev. 8, 485] the author proves: if an oval with a center  $O$  has the property that no concentric ellipse intersects it in more than four pairs of points symmetric with respect to  $O$ , then it has exactly two pairs of conjugate diameters. *W. Fenchel* (Copenhagen).

Vell', G. [Weyl, H.]. *On the determination of a closed convex surface by its line element.* Uspehi Matem. Nauk (N.S.) 3, no. 2(24), 159-190 (1948). (Russian)  
Translated from *Vierteljsschr. Naturforsch. Ges. Zürich* 61, 40-72 (1916).

Borsuk, Karol. *Sur la courbure totale des courbes fermées.* Ann. Soc. Polon. Math. 20 (1947), 251-265 (1948).

The reviewer [Math. Ann. 101, 238-252 (1929)] proved for  $n=3$  that the total curvature of a closed curve in  $n$ -space is greater than or equal to  $2\pi$  with equality only in the case of a plane convex curve. By first proving the corresponding inequality for closed polygons the author gets this result for an arbitrary value of  $n$  in a new way. [For one of the lemmas, stating a necessary and sufficient condition that a closed curve on the unit sphere be the tangent indicatrix of a closed curve, see W. Fenchel, Jber. Deutsch. Math. Verein. 39, 183-185 (1930); also M. Vigodsky, Rec. Math. [Mat. Sbornik] N.S. 16(58), 73-80 (1945); these Rev. 7, 75. It may be mentioned that B. Segre, Boll. Un. Mat. Ital. 13, 279-283 (1934), has sharpened the above theorem considerably and observed that his elementary and simple proof holds for an arbitrary value of  $n$ .] The author concludes by stating the problem of whether the total curvature of a knotted curve is greater than or equal to  $4\pi$ .

*W. Fenchel* (Copenhagen).

### Algebraic Geometry

Néron, André. *Une propriété arithmétique des faisceaux linéaires de courbes de genre 1.* C. R. Acad. Sci. Paris 226, 1781-1783 (1948).

Let us consider a finite algebraic field  $k$ , and choose arbitrarily on a plane  $(x, y)$  8 points over  $k$  such that no 3 of them are on a line and no 6 of them are on a conic. If  $\varphi(\lambda)$  denotes any rational function over  $k$  and  $f(x, y)$ ,  $g(x, y)$  are two independent cubic polynomials over  $k$  vanishing at the 8 points, then there is an infinity of values of  $\lambda$  in  $k$  such that the plane cubic  $f(x, y) + \varphi(\lambda)g(x, y) = 0$  has rank  $r \geq 8$  over  $k$ . The proof of this theorem is only sketched, and consists of two parts, one geometric and the other arithmetic in character. The first part deals with some covariant curves of a pencil of plane cubics and leads also to other geometric results. The second part is based on the following theorem, which is only stated. Let  $E$  be a finite set of plane algebraic curves over  $k$  irreducible and of genus greater than or equal to 1, and let  $\Phi(x, y, \lambda)$  be a polynomial in  $x, y$  whose coefficients are rational functions of  $\lambda$  over  $k$ ; then there is an infinity of values of  $\lambda$  in  $k$  for each of which no point over  $k$  lies simultaneously on  $E$  and on the curve  $\Phi(x, y, \lambda) = 0$ . [Reviewer's remark. One has obviously to disregard the points over  $k$ , if any, lying on  $E$  and on the curve  $\Phi(x, y, \lambda) = 0$  for all  $\lambda$  of  $k$ .] *B. Segre.*

Terracini, Alejandro. *On some classes of skew rational curves.* Univ. Nac. Tucumán. Revista A. 6, 167-186 (1947). (Spanish)

Dans ce mémoire l'auteur revient, pour l'approfondir et la compléter, sur une étude antérieure [Rend. Circ. Mat. Palermo 56, 112-143 (1932); travail désigné par (P) dans l'étude actuelle] consacrée aux correspondances algébriques  $\Gamma$  obtenues en se donnant une courbe algébrique quelconque  $C^n$  d'ordre  $n$  de l'espace ordinaire et en associant, à un point quelconque  $P$  de  $C^n$ , l'un quelconque des  $n-3$  points  $P'$  où le plan osculateur en  $P$  coupe à nouveau la courbe. Pour des  $C^n$  particulières  $\Gamma$  peut être décomposée en correspondances partielles dont certaines peuvent être des projectivités (correspondances algébriques (1, 1) sur des courbes rationnelles), ces projectivités pouvant se réduire à des homographies (induites sur  $C^n$  par des homographies de l'espace ambiant). En (P) se trouve posé le problème de la recherche de toutes les  $C^n$  rationnelles pour lesquelles  $\Gamma$  se décompose en le nombre maximum  $n-3$  d'homographies, et il y est dit, par inadvertance, que les seules solutions sont fournies par les courbes  $W$  de Lie  $[x_0:x_1:x_2:x_3=t^n:t^n:t^n:1]$  et la  $C^4$  d'Egan  $[x_0:x_1:x_2:x_3=2t^6-1:t^6-2t^6:t:t^6]$ . A ces courbes il convient d'ajouter la  $C^7$   $[x_0:x_1:x_2:x_3=t^4:t^4:t^7-2t:1-2t^6]$ .

Le travail débute par le rappel de certains résultats établis en (P), se rapportant au cas où la correspondance  $\Gamma$  relative à une  $C^n$  (non  $W$ ) admet au moins une composante homographique. Une telle composante est alors nécessairement involutive et est induite par une homographie biaxiale harmonique  $\Omega$ . La  $C^n$  est asymptotique rationnelle d'une surface réglée algébrique rationnelle appartenant à une congruence linéaire dont les directrices sont les axes de  $\Omega$ , et, comme telle, appartient à l'un des  $\infty^1$  complexes linéaires contenant la congruence. De là résulte un procédé algébrique de détermination des  $C^n$  admettant au moins une composante homographique pour les  $\Gamma$  associées, consistant essentiellement à exprimer la nullité d'un certain déterminant d'ordre  $n-2$  dont les éléments sont linéaires et homogènes par rapport aux coefficients de deux des polynômes  $x_i(t)$  définissant  $C^n$ . La mise en oeuvre du procédé auquel il vient

d'être fait allusion, dans le cas où il existe  $s$  ( $2 \leq s < n-3$ ) homographies composantes pour  $\Gamma$ , exige l'introduction du groupe d'ordre fini  $G_s$  des homographies de l'espace engendrées par les  $\Omega_i$  ( $i=1, 2, \dots, s$ ) attachées aux diverses composantes de  $\Gamma$ , et du groupe  $G_s$  (holoédriquement isomorphe à  $G_s$ ) des projectivités induites par  $G_s$  sur  $C^n$ , groupe engendré par les involutions  $I_i$  induites par les  $\Omega_i$  sur  $C^n$  et par rapport auquel l'ensemble  $\{I_i\}$  des  $I_i$  est invariant. Se basant sur le fait que  $\{I_i\}$  ne peut être que l'un des ensembles  $M, M', \dots$  d'involutions de  $G_s$ , formés chacun d'éléments équivalents dans  $G_s$ , et sur la remarque qu'il ne peut y avoir trois  $I_i$  formant avec l'identité un  $G_4$  quadrinvolutif, l'auteur discute les différentes possibilités pour  $G_s$ . Il montre que les  $G_s$  cycliques ainsi que le  $G_{24}$  tétraédrique et le  $G_{24}$  icosaédrique sont inadmissibles. Pour un  $G_{24}$  octaédrique, les ensembles  $M, M', \dots$  d'involutions signalés plus haut se réduisent à deux  $[M, M']$  formés respectivement de 3 et 6 involutions, et  $\{I_i\}$  coïncide nécessairement avec  $M'$ . Enfin la considération des  $G_{24}$  diédriques conduit, en dehors du cas  $[h=2, s=2]$ , aux deux hypothèses suivantes: (a)  $h$  étant de parité quelconque,  $\{I_i\}$  est formé des  $2h$  involutions de  $G_{24}$  (involution invariante exclue); (b)  $h=2 \pmod{4}$ , les  $M, M', \dots$  se réduisent alors à trois ( $M$  et  $M'$  étant formés de  $\frac{1}{2}h$  involutions et  $M''$  de l'involution invariante), et  $\{I_i\}$  est constitué par  $[M, M', M'']$ . L'application des résultats énoncés au cas  $s=n-3$  conduit à ne retenir que la possibilité d'un  $G_{24}$  diédrique. Le cas  $h=2$  donne la  $C^*$  d'Egan, l'hypothèse (a) ci-dessus est inadmissible, et (b) donne l'unique courbe  $C^*$  annoncée au début de cette analyse.

Le mémoire se termine par une étude rapide du cas  $2 \leq s < n-3$ . L'auteur montre que les  $C^*$  peuvent alors appartenir à deux espèces distinctes caractérisées par les circonstances suivantes: pour la première espèce deux des  $s$  composantes de  $\Gamma$  sont permutables; pour la deuxième espèce  $\Gamma$  admet  $r$  composantes coïncidant avec les  $r$  involutions d'un  $G_3$ , diédrique ( $r$  impair). Il établit que pour les  $C^*$  de première espèce  $n$  est impair et indique comment on peut obtenir les  $C^*$  pour  $n \geq 7$ . Il montre enfin que, pour chaque valeur impaire de  $r$  première avec  $n$  et telle que  $n > 2r$ , il existe des courbes  $C^*$  de deuxième espèce et indique le moyen de former leurs équations.

P. Vincensini (Besançon).

**Drach, Jules.** Détermination des lignes d'osculation quadrique (lignes de Darboux) sur les surfaces cubiques. Lignes asymptotiques de la surface de Bioche. C. R. Acad. Sci. Paris 226, 1561-1564 (1948).

Continuation of a preceding paper [same C. R. 224, 309-312 (1947); these Rev. 8, 6] on Darboux curves. The main result is that these curves are represented on all cubic surfaces by an equation  $v - ju = \text{constant}$ , where  $j^2 = 1$ ,  $v$  and  $u$  being suitably defined asymptotic parameters. Examples of integration of the differential equations for particular surfaces.

E. Bompiani (Rome).

**Gaeta, Federico.** On the arithmetically normal surfaces and varieties of  $S_p$ . Revista Mat. Hisp.-Amer. (4) 8, 72-82 (1948). (Spanish)

L'auteur commence par rappeler que si la surface réductible  $F+F'$  ( $F, F'$  irréductibles et sans lignes multiples) est l'intersection complète de  $r-2$  formes de  $S_p$ , d'ordres  $n_1, n_2, \dots, n_{r-2}$ , si l'on désigne par  $\Delta_i, \Delta'_i$  les déficits des systèmes que déterminent les formes d'ordre  $l$  sur  $F$  et  $F'$ ,

par  $p_g, p_a; p_g', p_a'$  les genres géométriques et arithmétiques des deux surfaces, et si l'on pose  $\rho = \sum n_i - r - 1$ , on a  $\Delta'_i = p_g - p_a$  et  $\Delta_g = p_g' - p_a'$ . De là résulte que pour que  $F'$  soit régulière ( $p_g = p_g'$ ) il faut et il suffit que les formes d'ordre  $\rho$  ( $\rho > 0$ ) déterminent sur  $F$  un système linéaire complet, puis un critère de suffisance pour que, si  $F$  est arithmétiquement normale (si  $\Delta_g = 0$  pour  $1, 2, \dots$ ), toute  $F'$  complémentaire irréductible et sans points doubles le soit aussi: il faut et il suffit pour cela que  $F'$  soit régulière. Une condition nécessaire et suffisante pour qu'une surface irréductible régulière soit arithmétiquement normale est d'ailleurs que la section hyperplane générique le soit. Ces propositions conduisent à la suivante relative à  $S_4$  dont la réciproque est due à Severi. Une surface irréductible  $F$  de  $S_4$  régulière, subcanonique et arithmétiquement normale est l'intersection complète de deux formes et réciproquement. L'application d'un résultat antérieur de l'auteur sur les courbes gauches de résiduel un montre en outre qu'une condition nécessaire et suffisante pour qu'une  $F$  irréductible et régulière de  $S_4$  soit de résiduel un est que l'idéal de polynome correspondant à  $F$  admette une base minima formée de trois polynomes.

Passant ensuite aux variétés à  $d$  dimensions de  $S_p$ , l'auteur montre qu'étant données deux telles variétés irréductibles et sans singularités,  $V_d$  et  $W_d$ , constituant l'intersection complète de  $r-d$  formes d'ordres  $n_1, n_2, \dots, n-d$ , l' $H$ -idéal  $Z$  correspondant à la variété  $Z_{d-1} = (V_d, W_d)$  est la somme des  $H$ -idéaux  $V, W$  relatifs à  $V_d$  et  $W_d$ , d'où résulte que si  $V$  et  $W$  sont de première espèce  $Z$  l'est aussi, et que si  $Z = (V, W)$  est dépourvue de singularités elle est arithmétiquement normale si  $V$  et  $W$  le sont, cette dernière propriété admettant une réciproque établie pour  $d \geq 2$ . Après l'étude des courbes subcanoniques arithmétiquement normales de  $S_4$ , l'article se termine par la démonstration d'une propriété caractéristique des  $V_d$  arithmétiquement normales irréductibles sans singularités. Soit  $W_d$  une variété irréductible intersection ultérieure de  $r-d$  formes  $f_1 = f_2 = \dots = f_{r-d} = 0$  d'ordres  $n_1, n_2, \dots, n-d$ , et  $f_{r-d+1} = 0$  une forme d'ordre  $n_r - d + 1$  passant par  $W_d$  et coupant ultérieurement  $V_d$  suivant  $H_{d-1}$  en dehors de  $\Gamma_{d-1} = (V_d, W_d)$ . L'idéal homogène  $A$  constitutif par toutes les formes  $f$  telles que  $f = 0$  passe par  $W_d + H_{d-1}$  est engendré par les  $r-d+1$  formes  $f_1, f_2, \dots, f_{r-d+1}$  ( $A = (f_1, f_2, \dots, f_{r-d+1})$ ).

P. Vincensini (Besançon).

**Fano, Gino.** Nuove ricerche sulle varietà algebriche a tre dimensioni a curve sezioni canoniche. Pont. Acad. Sci. Acta 9, 163-167 (1945).

This note does not differ essentially from a note by the same author which was published a year later [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 436-466 (1946); these Rev. 8, 339]. It announces a proof of the irrationality of the general  $V_3^3$  in  $S_4$  and gives some indication of the nature of the proof. The complete proof is to appear in a memoir which is in course of publication. [In the review referred to above it was reported that the author has proved the irrationality of the varieties  $M_3^{p-2}$  in  $S_{p+1}$  (having canonical curve sections) for  $p=3, 4, 5, 6, 8, 13$  and the rationality of these varieties for all other values of  $p$ . From the present note it appears that this result refers only to those  $M_3^{p-2}$  on which every surface is a complete intersection (and hence, in particular, to those  $M_3^{14}$  in  $S_4$  which are birationally equivalent to the general  $V_3^3$  in  $S_4$ .)]

O. Zariski (Cambridge, Mass.).

**Godeaux, Lucien.** Étude d'une involution cyclique appartenant à une surface algébrique. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iași] 3, 107–112 (1948).

**Metelka, Josef.** On certain finite groups constructed from Cremona transformations of the 1st to 5th degrees. Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodnověd. 1946, 29 pp. (1947). (Czech)

1947

### Differential Geometry

**Beckenbach, E. F.** Some convexity properties of surfaces of negative curvature. Amer. Math. Monthly 55, 285–301 (1948).

If suitable "geodesic" parameters are chosen the first fundamental form of an analytic surface  $S$  assumes the form  $ds^2 = du^2 + \mu^2 dv^2$ ,  $\mu = \mu(u, v)$ . A necessary and sufficient condition that  $S$  is of nonpositive Gaussian curvature  $K$  is that  $\mu(u, v_0)$  is a convex function of  $u$  for each fixed  $v_0$ . From this remark the author derives a series of theorems concerning convexity and monotonicity properties of certain functions connected with geometrical figures on the surface. As typical examples the following may be mentioned. Let  $l(r)$  denote the length of the circumference of the geodesic circle with a fixed center and geodesic radius  $r$ . Then  $l(r) - 2\pi r$  is a nondecreasing convex function of  $r$ , and  $l(r)/(2\pi r)$  is nondecreasing. If the surface is not developable, these functions are strictly convex and increasing, respectively. If  $a(r)$  denotes the area of this geodesic circle, the corresponding functions  $a(r) - \pi r^2$  and  $a(r)/(2\pi r^2)$  have the same properties. In some cases analogous results are obtained for surfaces with  $K \geq 0$ , where "convex," "non-decreasing" and "increasing" are to be replaced by "concave," "nonincreasing" and "decreasing," respectively. From this it follows that these properties are characteristic for the surfaces of negative (positive) curvature.

W. Fenchel (Copenhagen).

**Jaspar, S. J.** Helices in a flat space of four dimensions. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iași] 2, 262–267 (1947).

The parametric equations of the helix can be written in the form

$$\begin{aligned} x_1 &= (c/M) \cos Ms, & x_3 &= (E/N) \cos Ns, \\ x_2 &= (c/M) \sin Ms, & x_4 &= (E/N) \sin Ns, \end{aligned}$$

where  $C, E, M$  and  $N$  are constants, the latter two specific functions of the three positive, constant, nonzero curvatures. The helix can be represented by two moving points, one moving along the circle which is the  $x_1 x_3$  projection of the helix, the other moving along the  $x_2 x_4$  projection, and the radii and angular velocities for these circles are presented in tabular form for different curvature possibilities.

A. Schwartz (New York, N. Y.).

**Schafer, Alice T.** The neighborhood of an undulation point on a space curve. Amer. J. Math. 70, 351–363 (1948).

Un point d'ondulation est un point où la courbe  $(C)$  à un contact du troisième ordre avec sa tangente. Dans le § 1 du travail, un repère projectif  $OABC$  est déterminé en un tel point  $O$  par les conditions suivantes: (a) le point  $O$  est justement le point d'ondulation de  $(C)$ ; (b)  $OA$  est

tangente à  $(C)$ ; (c) le plan  $OAB$  a un contact du quatrième ordre avec  $(C)$ ; (d) il y a une surface cubique réglée et une seule qui a un contact du sixième ordre avec la courbe et dont  $OA$  est une génératrice double; des conditions sont imposées au repère qui reviennent à le prendre de manière que cette surface cubique ait pour équation (1)  $2x_2x_3^2 - 2x_2x_3x_4 + x_2x_4^2 - 2x_3x_4 = 0$ . Dans ce repère on passe aux coordonnées non homogènes ( $x = x_2/x_1$ ,  $y = x_3/x_1$ ,  $z = x_4/x_1$ ) et on écrit les développements de  $y, z$  en fonctions de  $x$  autour de  $O$  jusqu'aux termes du treizième degré.

On étudie ensuite les cônes du deuxième et du troisième degré osculateurs à la courbe (ce sont des plans multiples), la quadrique osculatrice (un plan double), la cubique osculatrice (1) et enfin le cône du quatrième degré osculateur. On étudie les intersections de la développable des tangentes avec les plans de coordonnées du repère et les projections de  $(C)$  sur les plans de coordonnées, le centre de projection étant un point quelconque de l'espace ou bien des points ayant certaines propriétés spéciales. M. Haimovici.

**Hsiung, Chuan-Chih.** Differential geometry of a surface at a parabolic point. Amer. J. Math. 70, 333–344 (1948).

Cet article entreprend l'étude de la géométrie projective différentielle d'une surface en un point parabolique: point où les deux tangentes asymptotiques coïncident. La section tangentielle, c'est-à-dire la section de la surface par le plan tangent, présente un point de rebroussement ou un point tacnodal. Cinq cas peuvent se présenter suivant que la section tangentielle admet un point de rebroussement, ou un point tacnodal simple, ou un point tacnodal à simple ou à double inflexion, ou un point tacnodal symétrique. Pour chacun de ces cas, l'auteur donne une forme canonique du développement en série de la surface au voisinage du point parabolique et il donne aussi l'interprétation géométrique du système de référence correspondant. M. Decuyper.

**Rollero, Aldo.** Considerazioni sull'intorno di un punto parabolico di una superficie. Matematiche, Catania 2, 13–17 (1946).

Let  $O$  be a parabolic point of a surface  $S$ . The main purpose of this paper is to discuss the type of singularity  $O$  may be of the asymptotic curve  $C$  through  $O$ . The main results are embodied in the following theorems. The parabolic point  $O$  is ordinarily a simple cusp of  $C$ . Every plane through the asymptotic tangent at  $O$  has contact of the fifth order with  $C$ . The projection  $C'$  of  $C$  from an arbitrary point  $P$  on such a plane has a cusp of the first kind at  $O$ , unless  $P$  is in the tangent plane to  $S$  at  $O$  (but not on the asymptotic tangent) in which case  $C'$  has a branch of the second order and fourth class at  $O$ . V. G. Grove.

**Rollero, Aldo.** Un'osservazione sulle tangenti di Darboux e di Segre. Matematiche, Catania 2, 18–19 (1946).

Given two tangents  $t_1, t_2$  to a surface  $S$  at a nonparabolic point  $O$ , the unique conjugate tangents separating  $t_1, t_2$  harmonically are called by the author principal lines relative to  $t_1, t_2$  [called the associate conjugate tangents of  $t_1, t_2$  by G. M. Green, Trans. Amer. Math. Soc. 21, 207–236 (1920)]. The purpose of this note is to call attention to the fact that if  $t_1, t_2$  coincide with tangents of Darboux (or Segre) their principal lines are the third tangent of Darboux (or Segre) and the corresponding conjugate tangent of Segre (or Darboux). V. G. Grove (East Lansing, Mich.).

**Myller, A.** *Cordes orthoptiques.* Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iași] 3, 95–97 (1948). (Romanian. French summary).

L'auteur étudie les courbes dont les cordes vues sous un angle droit d'un point fixe  $O$  ont des propriétés telles que: la longueur constante; forment avec le point  $O$  des triangles d'aire, de périmètre, d'angles constants, etc. On obtient des équations fonctionnelles dont on donne les solutions.

*Author's summary.*

**Salini, Ugo.** *Le curve "S" del piano affine.* Matematiche, Catania 1, 123–146 (1946).

The parameter  $t$  of a curve  $C$  in the affine plane with equations  $x=x(t)$ ,  $y=y(t)$  may be so chosen that the differential equation of  $C$  may be written in the form  $\theta''' + r\theta' = 0$ , and  $|x', y''| = 1$ . The function  $r$  and parameter  $t$  are affine invariants of  $C$ . The function  $r$  is called the affine curvature of  $C$ . There is a pencil of central conics having four point contact with  $C$  at one of its points  $O$ . The centers of these conics lie on a line called the affine normal. The focal point of the normal at  $O$  is the center of affine curvature. A line parallel to the tangent to  $C$  at  $O$  intersects  $C$  in two points, the locus of whose midpoint is a curve  $C'$  called by the author the satellite curve of  $C$ . Among the pencil of conics having four-point contact with  $C'$  at  $O$  is a parabola called the second principal parabola of  $C$  at  $O$ . The remainder of the paper is devoted to nodal cubics having 7- and 8-point contact with  $C$  at  $O$ , and to special curves  $C$  for which the tangent at  $O$  coincides with other covariant lines associated with  $C$ . *V. G. Grove* (East Lansing, Mich.).

**Popa, Ilie.** *Sur une propriété caractéristique commune aux cercles plans, aux cercles géodésiques et aux surfaces à courbure moyenne constante.* C. R. Acad. Sci. Paris 226, 2120–2122 (1948).

Let  $\mathbf{r} = \mathbf{r}(s)$  be the vector equation of a plane curve  $C$  and  $\mathbf{e}(s)$  a differentiable vector function of  $s$ . The curve  $C_1$ , defined by  $\mathbf{r}_1 = \mathbf{r} + \mathbf{e}$ , higher powers of  $|\mathbf{e}|$  being neglected, is called an infinitesimal transform of  $C$ . If  $C$  is closed, of length  $L$  and area  $A$ ,  $\mathbf{e}(s)$  is periodic with amplitude  $L$  and  $C_1$  is closed, then a necessary and sufficient condition that there exists a linear relation between the area  $A_1$  and the length  $L_1$  of  $C_1$  is that  $C$  is a circle. This theorem is extended to a curve  $C$  on a regular surface  $S$ . In the extension  $C$  must be a geodesic circle on  $S$ , and,  $C$  being closed,  $S$  must have constant total curvature. An extension to infinitesimal deformations  $S_1$  of a surface  $S$  gives the theorem: a necessary and sufficient condition that there exists a linear relation between the area and volume of  $S_1$  is that  $S$  is a surface with constant mean curvature.

*V. G. Grove* (East Lansing, Mich.).

**Popa, Ilie.** *Transformations infinitésimales des surfaces conservant les aires.* Bull. École Polytech. Jassy [Bul. Politehn. Asachi. Iași] 3, 120–123 (1948).

If a surface is given by the vector equation  $\mathbf{x} = \mathbf{x}(u, v)$  and  $\mathbf{e} = \mathbf{e}(u, v)$  is an infinitesimal vector, the surface  $\mathbf{x}_1 = \mathbf{x} + \mathbf{e}$  may be said to be obtained from the original surface by an infinitesimal distortion. The author here investigates the conditions under which this is an area preserving transformation. A necessary and sufficient condition that it be area preserving is that the divergence of the vector  $\mathbf{e}$  shall vanish.

This implies that the curves whose lengths are unchanged form an orthogonal system and conversely.

*S. B. Jackson* (College Park, Md.).

**Wuyts, P.** *A characteristic property of developable ruled surfaces.* Simon Stevin 25, 224–227 (1947). (Dutch)

Let  $C$  be a curve on a developable ruled surface. The planes through the points of  $C$  normal to the generators form a one-parameter family. The characteristic lines are parallel to the normal planes of  $C$ . The author proves the converse theorem: if a ruled surface contains a curve  $C$  with the property mentioned above, it is a developable surface.

*J. Haantjes* (Amsterdam).

**Lalan, Victor.** *Les surfaces à courbure moyenne isotherme et le problème d'Ossian Bonnet.* C. R. Acad. Sci. Paris 226, 1950–1952 (1948).

It is proved that the only surfaces for which the lines of constant mean curvature are simultaneously isothermals and geodesic parallels are surfaces admitting a one-parameter group of motions or are surfaces of O. Bonnet of the third class. This result is a partial converse of the known theorem: on a surface of O. Bonnet of the third class the lines of constant mean curvature are both isothermals and geodesic parallels. *C. B. Allendoerfer* (Princeton, N. J.).

**Backes, Fernand.** *Sur des familles de congruences W.* C. R. Acad. Sci. Paris 226, 1952–1954 (1948).

Let four conjugate nets of a periodic sequence of Laplace of period four be generated by points  $x_1, x_2, x_3, x_4$ . The diagonals  $x_1x_2, x_3x_4$  are known to generate  $W$ -congruences. It is shown that functions  $f, g, p, q$  exist so that the lines  $y_1y_2, y_3y_4$  determined by the points  $y_1 = x_1 + fx_3, y_2 = x_2 + gx_4, y_3 = px_2 + x_3, y_4 = qx_1 + x_4$  also generate  $W$ -congruences. That is, such a periodic sequence of Laplace gives rise to a quadruple infinity of  $W$ -congruences.

*V. G. Grove.*

**Backes, Fernand.** *Un cas de congruences doublement stratifiables.* C. R. Acad. Sci. Paris 227, 257–258 (1948).

This note is a continuation of an earlier paper [see the preceding review]. The question is raised of the existence of a transversal surface for each of a certain pair of congruences, discussed in the earlier paper, whose tangent plane at its intersection with the line of one congruence contains the line of the other congruence. The condition is expressed as the vanishing of one of the Darboux invariants of a conjugate net appearing in the discussion. *V. G. Grove.*

**Marcus, F.** *Sur quelques surfaces en relation avec les congruences de Waelsh.* Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iași] 2, 270–272 (1947).

A congruence is called a congruence of Waelsh if the asymptotic net on each focal surface corresponds to a conjugate net on the other. The congruence is taken to be the congruence of tangents to the curves of a conjugate net on a surface  $S$  referred to the asymptotic net. Conditions that the parametric net on the other focal surface be a conjugate net are found. Four cases arise. In three of these cases the conjugate net on  $S$  necessarily has equal point invariants, and in the fourth the associate conjugate net has equal Darboux invariants. *V. G. Grove* (East Lansing, Mich.).

**Marcus, F.** *Sur les réseaux et surfaces R.* Mathematica, Timișoara 23, 129–130 (1948).

A conjugate net is said to be an  $R$ -net if the congruences of the tangents to its curves are  $W$ -congruences. It is stated

without proof that the defining differential equations of the sustaining surface may be written in one of five forms. One of these forms implies that that surface is a quadric.

V. G. Grove (East Lansing, Mich.).

**Marcus, F.** On the converse of Bianchi's permutability theorem. *Ann. of Math.* (2) 49, 710-713 (1948).

L'auteur donne une démonstration de la réciproque du théorème de permutabilité de Bianchi pour les congruences  $W$ . Soient  $(M_0)$ ,  $(M_1)$ ,  $(M_2)$ ,  $(M_3)$  quatre surfaces, nappes focales de quatre congruences rectilignes en relation conforme au théorème de permutabilité (la congruence  $(M_i M_j)$  admet  $(M_i)$  et  $(M_j)$  pour nappes focales). S'il existe  $\infty^1$  surfaces  $(M_k)$  ( $M_k = M_1 + p_k M_3$ ) telles que les congruences  $(M_k M_0)$ ,  $(M_k M_2)$ , admettent respectivement pour nappes focales  $(M_0)$ ,  $(M_2)$  et  $(M_3)$ , toutes les congruences  $(M_k M_0)$  et  $(M_k M_3)$  sont  $W$ . La démonstration repose sur la méthode du repère mobile  $(M_0 M_1 M_2 M_3)$  d'É. Cartan.

P. Vincensini (Besançon).

**Haimovici, Adolf.** Sur une famille de surfaces en relation avec les développables d'une congruence de droites et sur les surfaces à réseau orthogonal de courbes planes. *Bull. École Polytech. Jassy* [Bul. Politehn. Gh. Asachi. Iași] 3, 61-77 (1948).

The purpose of this paper is to study the class of surfaces whose lines of curvature are plane curves, or more generally to find all surfaces which are cut by the developables of a congruence in an orthogonal net. Among the results found the following may be quoted. If an arbitrary congruence  $G$  is given, then through each point in space there passes a surface on which the developables of  $G$  correspond to an orthogonal net. If a given congruence  $G$  has developable focal surfaces, then through a given curve in space there passes a unique surface on which the tangent planes to the focal surfaces of  $G$  determine an orthogonal net of plane curves. The method used is the classical method. Special attention is paid to congruences with degenerate focal surfaces.

V. G. Grove (East Lansing, Mich.).

**Vasil'ev, A. M.** Involutory systems of line complexes. *Doklady Akad. Nauk SSSR* (N.S.) 61, 189-191 (1948). (Russian)

The  $\infty^4$  lines of a  $P_3$  may be mapped onto a hyperquadric  $Q_4$  in a  $P_5$ . To a linear complex of  $P_3$ , there corresponds a hyperplane of  $P_5$  which intersects  $Q_4$  in a  $V_3$ , the map of the complex. The  $\infty^8$  lines of an arbitrary complex  $C_3$  map onto a  $V_3'$  of  $Q_4$ . Each line of  $C_3$  belongs to a pencil of tangent linear complexes, that is, the correspondents of the pencil of hyperplanes on the tangent  $S_3$  of  $V_3'$ . A pair of complexes on a common line  $p$  will be said to intersect in involution along the line if each of the  $\infty^1$  linear complexes on  $p$  tangent to one of the pair is in involution with each of the  $\infty^1$  linear complexes on  $p$  tangent to the other of the pair. On a given line  $p$  there exist at most four complexes intersecting in involution along  $p$ . By an involutory system of complexes will be understood four one-parameter families of complexes with the property that any line common to all is likewise a line along which any pair of the complexes intersect in involution. Differential equations defining an involutory system are set up by means of a moving reference frame and the method of exterior differentiation of Cartan. Differential properties of the system are read from the equations, among which is the result that the four complexes of the system intersect in pairs to form  $W$ -congruences.

T. C. Doyle (Hanover, N. H.).

**Vincensini, Paul.** Questions de géométrie liées au caractère invariant de certains réseaux par déformation arbitraire de leur surface support. *Ann. Sci. École Norm. Sup.* (3) 64 (1947), 197-226 (1948).

The main purpose of this paper is to discuss the following problem. Let  $S$  be an arbitrary surface, generated by a point  $P$ . Let  $I$  be an arbitrary point in the tangent plane to  $S$  at  $P$ . What are the curves  $C$  on which  $P$  must move in order that the locus of  $I$  shall be orthogonal to the locus of  $P$ , and remain so under arbitrary deformations of  $S$ ? It is found that the locus of  $P$  is any one of the curves of a conjugate net  $(D)$ . However, for certain positions of  $I$  this net may be indeterminate. In particular, if  $P$  describes any curve  $C$  on  $S$ , and  $I$  is the center of geodesic curvature of  $C$  at  $P$ , then the loci of  $P$  and  $I$  are orthogonal. In case the net  $(D)$  is indeterminate the first fundamental form of  $S$  may be written in the form  $ds^2 = [(\partial a / \partial u) du]^2 + a^2 dv^2$ ,  $a = PI$  being the radius of geodesic curvature of the curve  $u = \text{constant}$ . The surfaces  $S$  are those applicable to the spiral surfaces of M. Levy. Conditions are found in order that  $(D)$  be a conjugate net or the asymptotic net. On all surfaces  $S$  there exists an infinity of conjugate nets  $(D)$ , depending on two arbitrary functions of one variable; on every surface the asymptotic net is a  $(D)$  net.

V. G. Grove (East Lansing, Mich.).

**Bompiani, E.** Determinazioni differenziali relative alle superficie di Veronese. *Pont. Acad. Sci. Acta* 8, 39-47 (1944).

L'auteur appelle calottes du deuxième ou du troisième ordre d'une surface d'un espace  $S_n$ , les voisinages du deuxième ou du troisième ordre d'un point de celle-ci, dit centre de la calotte. Étant donnée une calotte  $\sigma_2$  du deuxième ordre, de centre  $O$ , de  $S_n$ , il montre qu'il existe un ensemble de  $\infty^8$  surfaces de Veronese de  $S_n$  admettant cette calotte, et donne une représentation paramétrique de cet ensemble. Le contact du deuxième ordre, suivant  $\sigma_2$ , des surfaces de Veronese de l'ensemble précédent, n'entraîne pas la coïncidence des plans des coniques de ces surfaces issues de  $O$ . Cette coïncidence ne peut avoir lieu qu'exceptionnellement. Dans le cas général, si  $F$  est l'une des surfaces envisagées, il existe  $\infty^8$  surfaces  $F$  de la famille considérée qui ont en commun avec  $F$  les plans de trois coniques fixées issues de  $O$ , et, entre deux quelconques de ces  $\infty^8$  surfaces, il existe une projectivité bien déterminée conservant la calotte et les trois points d'intersection (autres que  $O$ ) des trois couples de coniques situées dans les mêmes plans. En ce qui concerne les calottes du troisième ordre, l'auteur montre qu'étant donné l'une,  $\sigma_3$ , de ces calottes sur une surface de Veronese  $F$ , il existe  $\infty^8$  surfaces de Veronese  $F$  admettant cette même calotte  $\sigma_3$ ,  $F$  et l'une quelconque des  $F$  associées ayant en commun deux coniques. Il met en évidence ce fait que, tandis qu'une  $\sigma_3$  d'une surface quelconque de Veronese est une calotte générique superficielle de  $S_n$ , il n'en est pas de même pour les  $\sigma_3$ , lesquelles doivent satisfaire à six relations particulières pour appartenir à une surface de Veronese. Alors qu'une surface quelconque de  $S_n$  peut, en l'un de ses points, être approximée jusqu'au deuxième ordre par  $\infty^8$  surfaces de Veronese, l'approximation jusqu'à l'ordre trois par de telles surfaces n'est généralement pas possible; cependant, s'il existe une surface de Veronese réalisant l'approximation du troisième ordre il en existe  $\infty^8$ .

Pour terminer l'auteur détermine les surfaces de Veronese qui ont en commun avec l'une d'elles deux calottes du

deuxième ordre à centres distincts; il montre qu'il existe  $\infty^1$  de ces surfaces, et que ces  $\infty^1$  surfaces sont tangentées tout le long de leur conique commune passant par les centres des deux calottes. En un point de cette conique commune autre que les deux centres, le contact de deux des  $\infty^1$  surfaces de Veronese précédentes est seulement du premier ordre, et, s'il est d'ordre deux, les deux surfaces coïncident.

P. Vincensini (Besançon).

**Chern, Shiing-Shen.** Sur les invariants de contact en géométrie projective différentielle. *Pont. Acad. Sci. Acta* 5, 123-140 (1941).

For the most part this paper is an exposition of the theory of the contact invariants of varieties lying in a projective space. The chief innovation is the systematic use of Cartan's "repère mobile" in developing the theory. In the case of two plane curves, the Smith-Mehmke invariant  $I$  is obtained and simple proofs are given of the theorems of C. Segre and of B. Segre which interpret  $I$ . For two curves in the  $n$ -dimensional projective space, the invariants of B. Segre and Su and the generalized Smith-Mehmke invariant are derived. Finally the case of two tangent surfaces in the three-dimensional projective space is considered. In addition to the two standard invariants, two new ones are obtained, and geometric interpretations analogous to those for  $I$  are derived for these.

C. B. Allendoerfer.

**Chern, Shiing-shen.** Sur une classe remarquable de variétés dans l'espace projectif à  $n$  dimensions. *Sci. Rep. Nat. Tsing Hua Univ.* 4, 328-336 (1947).

Let there be given a variety  $V_k$  in a projective space  $S_n$ . The  $n+1$  functions  $x^i(u^1, u^2, \dots, u^k)$  (none of the points  $(u)$  lying in the tangent space  $T_k$  at  $x$ ) satisfy differential equations of the form

$$\begin{aligned} \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} &= L_{\alpha\beta}^p \frac{\partial x^i}{\partial u^p} + p_{\alpha\beta} x^i + {}^{(s)}d_{\alpha\beta} {}^{(s)}y^i, \\ \frac{\partial {}^{(s)}y^i}{\partial u^\alpha} &= {}^{(s)}m_{\alpha}^p \frac{\partial x^i}{\partial u^p} + {}^{(s)}q_{\alpha} x^i + {}^{(s)}E {}^{(s)}y^i. \end{aligned}$$

The curves  $C$ ,  $u^\alpha = u^\alpha(i)$ , for which  $du^\alpha/dt = \lambda^\alpha$ , satisfy the equation  ${}^{(s)}\mu_{\alpha\beta} d_{\alpha\beta} \lambda^\alpha \lambda^\beta = 0$  and are said to form the asymptotic net on  $V_k$ . The curve  $C$  is such that its osculating plane lies in the linear space containing the space  $T_k$ . The variety  $V_k$  is called a variety of Cartan if it is possible to find a parametric representation of  $V_k$  such that the equations of the asymptotic net on  $V_k$  assumes the form  $\omega \mu (du^\alpha)^2 = 0$ . A variety  $V_k$  is said to possess a generalized conjugate net if on  $V_k$  there exist  $k$  families of curves, one and only one of each family through each point  $x$  of  $V_k$ , and such that the characteristic of the one-parameter family of spaces  $T_k$  along a curve of one of the families through  $x$  is tangent to all of the  $k-1$  other curves of the family through  $x$ . The main purpose of this paper is to prove the theorem: necessary and sufficient conditions that a variety be a variety of Cartan is that it sustains a generalized conjugate net, and that its osculating hyperplanes are of dimension  $2k$ . Generalized conjugate nets being so defined, the paper concludes with a discussion of generalized Laplace transforms of the generalized net. A variety of Cartan has  $k(k-1)$  transformations of Laplace, each of which is usually a variety of Cartan. The method used is that of moving frames of reference of Cartan.

V. G. Grove.

**Chern, Shiing-shen, and Wang, Hsien-chung.** Differential geometry in symplectic space. I. *Sci. Rep. Nat. Tsing Hua Univ.* 4, 453-477 (1947).

Les auteurs se proposent d'étudier les variétés ponctuelles plongées dans l'espace symplectique, c'est-à-dire dans un espace à  $2n+1$  dimensions admettant comme groupe fondamental le sous-groupe du groupe projectif laissant un système nul invariant (groupe simple à  $(n+1)(2n+3)$  paramètres). Si  $M$  et  $N$  désignent deux points analytiques de l'espace, l'équation du système nul est représentée par  $(MN)=0$ ; lorsqu'il en est ainsi, les points  $M$  et  $N$  sont dits conjugués. La méthode utilisée est celle d'É. Cartan; le papier débute par la détermination des repères symplectiques ( $2n+2$  points analytiques  $A_0, \dots, A_n, B_0, \dots, B_n$  astreints aux conditions  $(A_i A_j) = (B_i B_j) = (A_i B_j) = 0$  ( $i \neq j$ ),  $(A_0 B_0) = \dots = (A_n B_n) \neq 0$ ) et par celle des équations de structure, au sens de Cartan, du groupe symplectique. Une particularité intéressante de la géométrie symplectique est la non-équivalence des  $p$ -plans  $L_p$ , déterminés par  $p+1$  points  $M_0, \dots, M_p$ , indépendants. Chaque  $L_p$  admet un invariant arithmétique  $S$  qui est dit son rang et qui est la moitié du rang de la matrice antisymétrique

$$\begin{bmatrix} (M_0 M_0) & \dots & (M_0 M_p) \\ \dots & \dots & \dots \\ (M_p M_0) & \dots & (M_p M_p) \end{bmatrix}.$$

Il en est de même des  $p$ -éléments de contact  $(M, L_p)$  qui se différencient en outre en éléments du premier ou du second type selon que  $M$  est conjugué de tous les points de  $L_p$ , ou non. Les auteurs sont ainsi conduits à étudier, par la théorie des systèmes de Pfaff en involution, l'existence et le degré de généralité des variétés enveloppées par des éléments de contact d'un type donné. En particulier si une variété  $V_p$  ( $p \geq 2$ ) est enveloppée par une famille de  $p$ -éléments du premier type alors  $p \leq n$  et  $S=0$ . Le mémoire se termine par la détermination des formules de Frenet pour une courbe générale (courbe qui n'appartient à aucun sous espace plan et dont les éléments de contact osculateurs sont du second type et de rang maximum) et par les éléments d'une théorie des hypersurfaces  $V_{2n}$ .

A. Lichnerowicz (Strasbourg).

**Wang, Hsien-Chung.** Axiom of the plane in a general space of paths. *Ann. of Math.* (2) 49, 731-737 (1948).

Mit Hilfe des Begriffes der totalgeodätischen Fläche hat Cartan die dreidimensionale Riemannschen Räume konstanter Krümmung folgendermassen charakterisiert. Existiert im dreidimensionalen Riemannschen Raum in jedem Punkte zu jeder beliebigen Ebenenstellung eine totalgeodätische Fläche, dann ist der Raum von konstanter Krümmung. Verf. verallgemeinert diesen Satz für die allgemeinen Räume der Bahnen (paths). Das Gegenstück für die totalgeodätische Fläche wird dabei folgendermassen definiert. Eine  $k$ -dimensionale Mannigfaltigkeit  $S$  wird in einem Punkte  $P_0$  "eben" genannt (subspace flat at  $P_0$ ), wenn sämtliche Bahnen, die eine  $k$ -Stellung ( $k$ -dimensionaler Vektorraum) berühren,  $S$  angehören;  $S$  heißt "totaleben" (totally flat), wenn sie in jedem Punkte eben ist. Verf. bezeichnet die Voraussetzung, dass in jedem Punkt zu jeder  $k$ -Stellung eine totalebene  $k$ -dimensionale Mannigfaltigkeit existiert, als das Ebenenaxiom und beweist folgenden Satz. Ist in einem allgemeinen Raum der Bahnen das Ebenenaxiom erfüllt, dann ist der Raum projektiv-eben. Mit Hilfe dieses Satzes kann Verf. auch die projektive Ebenheit von "K-spreads" charakterisieren.

O. Varga (Debrecen).

**Rozenfel'd, B. A. The metric method in projective differential geometry and its conformal and contact analogues.** Mat. Sbornik N.S. 22(64), 457-492 (1948). (Russian)

The basic idea of the paper consists of establishing a correspondence between certain configurations in a given space and points of an appropriate metric space. The metric invariants of this space will then correspond to certain invariants of the original space, if the basic groups of the two are simply isomorphic. These ideas were successfully exploited by numerous geometers, particularly by Tzitzéica and Bompiani for the projective group and by Lie for the contact group. As a classic example the author gives the correspondence between lines in  $P_3$  (projective space of three dimensions) with points on a hyperquadric  $Q_4$  in  $P_4$ , the equation of  $Q_4$  being  $p^{01}p^{23} + p^{02}p^{13} + p^{03}p^{12} = 0$  (condition of simplicity),  $p^{ij}$  being the Grassmann coordinates of the line. In this case the projective group of  $P_3$  is simply isomorphic with the projective group of  $P_4$  leaving  $Q_4$  invariant. The author gives numerous examples of this metric method, such as the geometry of pairs of lines in  $P_3$ , certain linear congruences in  $P_3$ , as well as applications to conformal differential geometry, the conformal group of  $C_3$  being simply isomorphic with the projective group of  $P_4$  leaving a  $Q_3$  invariant. The last section of the paper deals with points, planes and spheres (oriented) of a  $C_3$  under contact transformations. To each "sphere" of  $C_3$  corresponds a point of  $P_4$  (using hexaspherical coordinates), the contact group of  $C_3$  being simply isomorphic to the projective group of  $P_4$  leaving a  $Q_4$  invariant. This group may be looked upon as the group of motions in  $P_4$ ,  $Q_4$  being the absolute, the metric of  $P_4$  in this case being doubly pseudoelliptic.

M. S. Knebelman (Pullman, Wash.).

**Rozenfel'd, B. A. Differential geometry of figures of symmetry.** Doklady Akad. Nauk SSSR (N.S.) 59, 1057-1060 (1948). (Russian)

In an earlier paper [same Doklady (N.S.) 57, 543-546 (1947); these Rev. 9, 249] the author has introduced figures of symmetry and considered spaces whose elements are such figures. The present note is devoted to differential geometry in such spaces. The basic concept here is that of a line element given by two infinitesimally close figures of symmetry; the invariants (or parameters) of such a line element are obtained from those of pairs of figures discussed in the cited note by a passage to the limit. Using the affine connection and the metric also introduced in that note it is possible to characterize a line element by a local vector which the author defines in terms of the Lie algebra of the basic group; he finds formulas connecting the components of the local vector with parameters and what he calls geometrical parameters of the line element.

The author states that in studying  $k$ -parameter families of figures of symmetry he has obtained formulas which express local parameters in terms of a direction in the family and which are generalizations of the formulas of Hamilton and Mannheim for congruences of lines in  $R$ , and are analogous to those of Coolidge for  $S_3$  and  ${}^1S_3$ . In order to obtain further generalizations of congruences certain figures of symmetry are designated as basic and "congruence" is defined as a family such that only one element of it is incident to such a basic figure. The local structure of a congruence is characterized by certain affinors, and under certain conditions the family in the large is determined by them (up to a transformation of the group). These affinors are special cases of those introduced by V. Wagner [Rec.

Math. [Mat. Sbornik] N.S. 10(52), 165-212 (1942); these Rev. 7, 33] and contain in turn several special cases considered by Dubnov and Coolidge. Special types of congruences may be characterized by imposing conditions on these affinors.

G. Y. Rainich (Ann Arbor, Mich.).

**Petrov, A. Z. On the curvature of Riemann spaces.** Doklady Akad. Nauk SSSR (N.S.) 61, 211-214 (1948). (Russian)

The author defines "quadratic" curvature for a Riemann space as a curvature determined by two divectors. It is a scalar invariant expressed in terms of the fundamental tensor and the Riemann-Christoffel tensor. The author proves that a space of constant curvature is also of constant quadratic curvature and that a  $V_4$  of constant quadratic curvature is of constant curvature. A  $V_4$  of constant quadratic curvature is an Einstein space. The author is apparently unaware of the reviewer's paper [Proc. Nat. Acad. Sci. U. S. A. 17, 43-47 (1931)], where similar definitions are given and more general theorems are proved.

M. S. Knebelman (Pullman, Wash.).

**Ruse, H. S. The self-polar Riemann complex for a  $V_4$ .** Proc. London Math. Soc. (2) 50, 75-106 (1948).

L'auteur se propose principalement d'étudier les directions principales de Struik [J. Math. Physics 7, 193-197 (1928)] d'une variété riemannienne  $V_4$  de signature quelconque, lorsque le complexe de Riemann de  $V_4$  est autopolaire par rapport à la quadrique fondamentale de la variété. Si  $S_3$  est l'espace projectif à l'infini de l'espace affine tangent à  $V_4$  en un point, l'équation du complexe de Riemann dans  $S_3$  est  $R_{ijkl}p^{ij}p^{kl}=0$ , où les  $p^{ij}$  sont les coordonnées plückériennes d'une droite de  $S_3$ . On sait [Ruse, Proc. Roy. Soc. Edinburgh. Sect. A. 62, 64-73 (1944); ces Rev. 6, 106] que ce complexe est autopolaire lorsque  $R^{ijkl}=\epsilon^i R^{ijkl}$ . Si  $\epsilon=+1$  (complexe autopolaire de première espèce),  $V_4$  est un espace d'Einstein, si  $\epsilon=-1$  (seconde espèce) on a en particulier  $R=0$ . La méthode d'étude utilise essentiellement les procédés de la géométrie projective classique et une représentation des complexes linéaires de  $S_3$  par les points d'un espace projectif  $S_4$ . Les complexes spéciaux correspondent aux points d'une quadrique de  $S_3$  de coefficients  $\epsilon_{ab}$ ; au complexe de Riemann et au complexe des droites de  $S_3$  qui sont tangentées à la quadrique fondamentale, correspondent dans  $S_4$  les quadriques de coefficients  $R_{ab}$  et  $g_{ab}$  ( $\alpha, \beta=1, \dots, 6$ ). L'auteur établit en particulier que, pour un complexe de Riemann autopolaire de première espèce, il est possible de choisir dans  $S_4$  un système de coordonnées tel que les équations des trois quadriques  $R_{ab}$ ,  $g_{ab}$ ,  $\epsilon_{ab}$  se réduisent à des sommes de carrés. On obtient ainsi dans l'espace six ensembles de "directions principales" orthogonales pourvu que les diviseurs élémentaires de la matrice  $[R_{ab} - \rho g_{ab}]$  soient simples. Ce sont six des quinze ensembles de directions principales de Struik. La réalité de ces directions principales orthogonales est étudiée pour les différentes signatures de la métrique de  $V_4$ . Dans le cas hyperbolique normal, il y a coïncidence avec les directions principales définies par Watson [Proc. Edinburgh Math. Soc. (2) 6, 12-16 (1939)] pour un champ gravitationnel. Une étude très complète des autres directions de Struik est faite.

A. Lichnerowicz (Strasbourg).

**Johnson, Paul B. The role of the directrix in Levi-Civita parallelism.** Duke Math. J. 15, 389-405 (1948).

Levi-Civita parallelism of a vector in an  $n$ -dimensional Riemannian space  $V_n$  depends upon the space and upon

the curve (directrix) along which the vector is displaced. The effects upon parallelism of infinitesimal closed directrices, or infinitesimal shifts in a particular directrix, are well known. The author develops a theory valid for finite directrices and shifts. The finite properties of Levi-Civita parallelism are shown to depend upon the matrizen functional. Formulas for the successive Fréchet differentials of the matrizen are obtained, leading to a generalized Taylor series expansion. Riemannian spaces are classified by means of Fréchet differentials of parallel displaced vectors with changes in directrix as increment. This classification divides all Riemannian spaces into two categories: flat spaces and all other spaces.

A. Fialkov (Brooklyn, N. Y.).

Verbičikil, L. L. Metric-differential characterization of hypersurfaces of the second order. *Doklady Akad. Nauk SSSR* (N.S.) 60, 1117-1118 (1948). (Russian)

Hypersurfaces of order two are characterized by the relation  $(n+1)K\pi_{ij,k} = K_{,i}\pi_{jk} + K_{,j}\pi_{ik} + K_{,k}\pi_{ij}$ , where  $\pi_{ij}$  are the coefficients of the second fundamental form, covariant differentiation is taken with respect to the first and  $K$  is total curvature. A tensor whose vanishing is equivalent to the above relation is used in proving that normality is a characteristic property of hyperquadrics, i.e., the property that for every family of lines of curvature there exists a family of  $(n-2)$ -dimensional surfaces orthogonal to them. Another characteristic property of hyperquadrics is stated as follows: along a line of curvature the corresponding principal curvature is proportional to the cube of each of the remaining principal curvatures.

G. Y. Rainich.

Laptev, G. F. The affine deformation of surfaces with preservation of the internal geometries. *Doklady Akad. Nauk SSSR* (N.S.) 58, 529-531 (1947). (Russian)

In an earlier note [C. R. (Doklady) Acad. Sci. URSS (N.S.) 41, 315-317 (1943); these Rev. 6, 107] the author

has defined internal geometries (or induced affine connections) of an  $n$ -dimensional surface embedded in an  $N$ -dimensional space with affine connection. He now calls two such surfaces applicable on each other if they can be mapped on each other with preservation of these internal geometries. He states that when the embedding space is affine the above notion of applicability coincides with applicability of order two as defined by E. Cartan [Congrès Internat. Math., Strasbourg, 1920, pp. 397-406]. A surface is considered deformable if it is applicable on another surface which cannot be obtained from it by a transformation of the affine group. Using Cartan's theory of Pfaffian systems the author arrives at the theorem: an  $n$ -dimensional surface in a  $N$ -dimensional affine space is deformable if  $n^2+n-2 \leq 2\rho$ , where  $\rho$  is the horizontal rank of a certain matrix and  $n^2+3n-2 \leq 2N$ ; in the case of inequality in the last relation any two surfaces are applicable on each other. He states also another theorem of this type involving the vertical rank of the same matrix.

G. Y. Rainich.

Norden, A. On normalized surfaces in a Möbius space. *Doklady Akad. Nauk SSSR* (N.S.) 61, 207-210 (1948). (Russian)

The general idea of normalized spaces was previously given by the author [Rec. Math. [Mat. Sbornik] N.S. 20(62), 263-281 (1947); these Rev. 9, 67]. In the present paper the normalizing curve is a circle passing through the point of the surface, the plane of the circle being orthogonal to the given surface. The pentaspherical coordinates of the tangent sphere containing the normalizing circle, as well as their partial derivatives (with respect to curvilinear coordinates of the surface) are expressed in terms of the intrinsic invariants of the surface. The author then gives 10 different types of normalizations and the nature of intrinsic geometry entailed by them.

M. S. Knebelman.

## NUMERICAL AND GRAPHICAL METHODS

Lane, W., and Sweeney, D. Table of Legendre polynomials  $P_n(\cos \theta)$  for  $N=0(1)20$ , and  $\theta=0^\circ(1)180^\circ$  to six decimals. United States Atomic Energy Commission, Oak Ridge, Tennessee MDDC-780, LADC-361. i+8 pp. 1947.

The first sentence of the text is, "The accompanying table of Legendre Polynomials was computed for use because no tables of the functions of the desired range were available from other sources." As R. C. Archibald has remarked [Math. Tables and Other Aids to Computation 3, 186 (1948)], a correct statement would have been, "What we have tried to do is to compute again a small section of a table readily available in many libraries [A. H. H. Tallqvist, Acta Soc. Sci. Fennicae. Nova Ser. A. 2, no. 11 (1938)]. We did not look at this table." The quoted review by Archibald points out numerous errors. R. P. Boas, Jr.

Turrell, F. M., and Vanselow, A. P. Tables of coefficients for estimating oblate and prolate spheroidal surfaces and volumes from spherical surfaces and volumes. For finding fruit surfaces and volumes. Proc. Amer. Soc. Hort. Sci. 48, 326-336 (1946).

The tables give the ratios,  $\sigma_o$ ,  $\sigma_p$ , and  $v_o$ ,  $v_p$ , respectively, of the surface and volume of an oblate or prolate spheroid to those of the smallest surrounding sphere (i.e., having

major axis as diameter). If  $\rho$  is the ratio of least to greatest diameter the functions tabulated are

$$(I) \quad \sigma_o = \frac{1}{2} + \frac{1}{4}\rho^2(1-\rho^2)^{-1} \log \frac{1+(1-\rho^2)^{\frac{1}{2}}}{1-(1-\rho^2)^{\frac{1}{2}}},$$

$$(II) \quad \sigma_p = \frac{1}{2}\rho^2 + \frac{1}{2}\rho(1-\rho^2)^{-1} \sin^{-1}(1-\rho^2)^{\frac{1}{2}},$$

$$(III) \quad v_o = \rho, \quad (IV) \quad v_p = \rho^2.$$

Three decimal values are given except for small  $\rho$  where a figure or two extra may be given. Graphs are also given. The paper deals with the application to fruit measurement, with numerical examples and a discussion of errors involved in using the tables. [Errors on p. 328 may cause confusion: (i) in fig. 1,  $d=2a$  should be the diameter, not the radius; (ii) the formulas at the foot of the page appear to give ratios of surface of spheroid to volume of sphere; in the first of them the factor before the log should be  $\rho^2/(1-\rho^2)^{\frac{1}{2}}$ .]

J. C. P. Miller (London).

Hartree, D. R., and Johnston, S. On a function associated with the logarithmic derivative of the gamma function. Quart. J. Mech. Appl. Math. 1, 29-34 (1948).

This tabulates the real part of  $\psi(k) - \log k + \frac{1}{2}k$ , where  $\psi(k) = \Gamma'(k)/\Gamma(k)$ , to 8 decimals, with first and second differences for  $k^2 = -1(0.01)1$ . When  $k^2 > 0$ , it is under-

stood that  $z > 0$ . Note that half the range of the table corresponds to pure imaginary  $z$ ; when  $z = ia$  ( $a > 0$ ), the function tabulated is the real part of  $\psi(ia) - \log a$ , that is,  $d\{\arg \Gamma(ia)\}/da - \log a$ . The imaginary part of  $\psi(ia)$  is  $-d \log |\Gamma(ia)|/da$ , that is, since  $|\Gamma(ia)| = (\pi a^{-1}/\sinh \pi a)^{1/2}$ , simply  $\frac{1}{2}\pi \coth \pi a + \frac{1}{2}a$ . *J. C. P. Miller* (London).

**Wilkes, M. V., and Renwick, W.** An ultrasonic memory unit for the EDSAC. *Electronic Engrg.* 20, 208-213 (1948).

The authors are engaged in the development and construction of an electronic digital computing instrument whose memory organ consists of a number of mercury delay lines similar to those being incorporated in the EDVAC. They point out that the central problem in the building of such a machine is the development of this memory unit, and they therefore devote the body of the paper to the engineering problems involved in this aspect of their researches. The mercury delay line consists of a tube filled with mercury and having piezo-electric crystals at either end. If a train of pulses is delivered to one of the crystals a corresponding train of ultrasonic waves will be propagated along the mercury column at a velocity of about 5000 feet per second. When these waves reach the other crystal they are converted back to electrical signals which can then be passed through an amplifier and returned to the first crystal. Thus it is possible to "remember" fairly large amounts of information in a dynamic storage organ. (In the EDSAC each line will hold 576 binary digits.) The pulses used in the EDSAC are modulated on a 13.5 Mc. per sec. carrier; each pulse is 0.9  $\mu$ sec. long and spaced about 2  $\mu$ sec. apart. Since the authors plan to control the frequency of their pulses by using one of their lines as a frequency standard, it suffices to maintain all their delay lines within  $\frac{1}{2}^\circ$  C of one another. In addition to a bank of long delay lines the authors also intend to use a number of small ones each of which has 18 or 36 pulse capacity. Evidently these will be used both in the control and computer organs of the EDSAC. The authors discuss not only the engineering of the delay lines but also a number of associated circuits used in their machine.

*H. H. Goldstine* (Princeton, N. J.).

**Lourye, A. L.** The reduction of round-off errors with the increase of the number of measurements. *Akad. Nauk SSSR. Prikl. Mat. Meh.* 11, 489-492 (1947). (Russian. English summary)

The author makes the deliberately paradoxical claim: "If the precision of a single measurement decreases, the mean measurement error tends towards zero." Actually the following is proved. Let  $\xi$  be the deviation of the actual from the ideal pointer position and let  $f(x, h)dx$  be the probability of the inequality  $x \leq \xi \leq x + dx$ . If  $f(x, h) = f(-x, h)$ , and decreases with increasing  $x$ , and if (1)  $\int_0^l f(x, h)dx \rightarrow 0$  when  $h \rightarrow 0$  for a certain fixed  $l$ , the mean measurement error tends toward zero with  $h$ .

The reviewer believes that the assumption (1) postulates that, for small  $h$ , the probability of a moderate deviation  $\xi$  ( $|\xi| < l$ ) is small. Since deviations are due to random causes, this implies that any departure whatever of the measuring setup from the ideal will cause a sizeable jump greater than  $l$  of the pointer. This surely requires a sensitive instrument and thus decreasing  $\int_0^l f(x, h)dx$  means increasing the precision of the instrument. If this interpretation is correct, the

author has misstated his result which, in its correct form, is rather obvious.

The phrase "round off" is mentioned in the title because scale readings are assumed to be multiples of the smallest scale division. *A. W. Wundheiler* (Chicago, Ill.).

**Marcantonio, Alessandro.** Saggio di un'applicazione del calcolo delle matrici alla teoria degli errori. *Pont. Acad. Sci. Acta* 10, 301-320 (1946).

**Greville, T. N. E.** Adjusted average graduation formulas of maximum smoothness. *Record. Amer. Inst. Actuar.* 36, 249-264 (1947).

The basis for this paper is the theory of Chebyshev's orthogonal polynomials and their application to adjusted average graduation formulas of maximum weight described by Lidstone [*J. Inst. Actuar.* 64, 128-159 (1933)]. It is the author's purpose to extend the application of these polynomials to formulas of maximum smoothness and to suggest an answer to the problem of how to deal with the end terms in an adjusted average graduation. In particular, he generalizes the results of W. F. Sheppard [*Proc. Fifth Internat. Congress Math.*, v. 2, Cambridge, 1913, pp. 347-384, in particular, pp. 365, 369] on graduation formulas of maximum weight and on the identity of the graduated value obtained by a formula of maximum weight with the value of the least-square polynomial. Practical applications are discussed. *E. Frank* (Chicago, Ill.).

**Greville, T. N. E.** Tables of coefficients in adjusted average graduation formulas of maximum smoothness. *Record. Amer. Inst. Actuar.* 37, 11-30 (1948).

This paper gives four-place tables of the coefficients in graduation formulas of maximum smoothness, which are worked out from formulas given in the paper reviewed above. The coefficients are given not only for the symmetrical formulas to be used over the major part of the sequence of values to be graduated but also for the unsymmetrical formulas to be employed near the ends of the sequence. Thus these tables make possible a graduation of the entire sequence of given values. *E. Frank* (Chicago, Ill.).

**Spiegelman, Mortimer, Wolfenden, Hugh H., and Greville, T. N. E.** Discussion: Adjusted average graduation formulas of maximum smoothness. *Record. Amer. Inst. Actuar.* 37, 31-36 (1948).

In their discussions, Spiegelman and Wolfenden comment on the advantages of the use of Tchebycheff's orthogonal polynomials for curve fitting by maximizing the reduction of error and for maximizing smoothness, as given by Greville [see the second preceding review]. In his reply, Greville gives a simplification of his formula [op. cit.] for the coefficients in a graduation formula of maximum smoothness.

*E. Frank* (Chicago, Ill.).

**Salzer, Herbert E.** Coefficients for complex quartic, quintic, and sextic interpolation within a square grid. *J. Math. Physics* 27, 136-156 (1948).

For interpolation in a square grid in the complex plane by quartic, quintic or sextic interpolation the author uses points lying in an L-shaped boundary, and he gives tables for the Lagrangian coefficients  $L_k(P)$  defined by  $f(z_0 + Ph) = \sum L_k(P) \cdot f(z_k)$ , to 5-9 decimal places.

*E. Bodewig* (The Hague).

Salzer, Herbert E. *Table of coefficients for obtaining the first derivative without differences*. National Bureau of Standards, Appl. Math. Ser., no. 2, 20 pp. (1948).

The differentiation of Lagrange's interpolation formula gives  $f'(a+ph) \approx (hc)^{-1} \sum_i C_i(p) f(a+ih)$ , where  $C_i(p)$  is a polynomial in  $p$  of degree  $n-2$  ( $n$  being the number of points). The author's table gives the exact values of the  $C_i(p)$ . For  $n=4, 5, 6$  the  $p$ 's are given at the interval 0.01; for  $n=7$  the interval is 0.1. *E. Bodewig* (The Hague).

Bodewig, E. *Sur la méthode de Laguerre pour l'approximation des racines de certaines équations algébriques et sur la critique d'Hermite*. Nederl. Akad. Wetensch., Proc. 49, 911-921 = Indagationes Math. 8, 570-580 (1946).

van der Corput, J. G. *Sur l'approximation de Laguerre des racines d'une équation qui a toutes ses racines réelles*. Nederl. Akad. Wetensch., Proc. 49, 922-929 = Indagationes Math. 8, 581-588 (1946).

These two papers furnish a timely modern discussion of Laguerre's method of approximation to the roots of an equation  $f(x)=0$  of degree  $n$  with only real roots [Oeuvres, v. 1, Paris, 1898, pp. 87-103] to which Hermite devoted a special article [Oeuvres de Laguerre, v. 1, pp. 461-468]. They may be read independently of each other and of previous writings. From the first paper, by Bodewig, the following three main points should be mentioned. (1) An attempt is made to reconstruct one essential point, suppressed by Laguerre, of the argument leading to the particular approximation employed. (2) It is shown that the repeated iteration of Laguerre's process

$$(1) \quad x_{k+1} = x_k - \frac{nf(x_k)}{f'(x_k) \pm [H(x_k)]^{\frac{1}{n}}},$$

$$H(x) = (n-1)f''(x) - n(n-1)f(x)f''(x),$$

converges cubically (at least) or only linearly according to whether the approximated root is simple or multiple. (3) It is shown that the iterated method of Newton will converge quadratically to a multiple root  $x$  of multiplicity  $m > 1$  provided that Newton's algorithm is modified to the new form  $x_{k+1} = x_k - mf(x_k)/f'(x_k)$ .

In the second paper, by van der Corput, the algorithm of Laguerre is generalized in a manner already suggested in Bodewig's paper. In (1)  $H(x_k)$  is replaced by  $\lambda H(x_k)$ , where  $\lambda = n - q/\lfloor (n-1)q \rfloor$  ( $q$  fixed = 1, 2, ...,  $n-1$ ). It is shown that if  $f(x)=0$  does have roots of multiplicity  $m$  ( $q \leq m < n$ ), then the process always converges to one such root and that the convergence is linear if  $m > q$  or at least cubic if  $m = q$ . Beyond this, the precise behavior of the sequence  $\{x_k\}$  as to monotonicity is described.

*I. J. Schoenberg* (Philadelphia, Pa.).

Kuntzmann, J. *Remarques sur le calcul approché des racines d'une équation*. Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.) 23, 143-144 (1948).

Let the zero of  $y=f(x)$  be required. Then when interpolating linearly between  $x=a$  and  $x=b$  the resulting  $x$  has an error of approximately  $f''(a)f_b/2f'_a$ , where  $c$  lies in the interval  $a, b$ . This holds also for  $a=b$ , that is, for the error in Newton's formula. This estimate is better than the usual one, but has the disadvantage of using the first and second derivatives. *E. Bodewig* (The Hague).

Hartman, Philip. *Newtonian approximations to a zero of a function*. Comment. Math. Helv. 21, 321-326 (1948).

The author investigates the convergence of the sequence  $x_{n+1} = x_n - f(x_n)/f'(x_n)$  when  $|f'(x)| \geq m > 0$  and proves some positive and some negative theorems, one of which is as follows. Let  $f(x)$  be defined on  $|x| \leq 1$ . Let  $f(0)=0$  and let  $f'(x)$  be continuous at  $x=0$ . Then the  $x_i$  will converge to 0 whenever  $|x_0|$  is sufficiently small. The negative theorems show that the continuity of the derivative  $f'(x)$  at the zero of  $f(x)$  cannot be omitted and further that the condition  $|f'(x)| \geq m$  cannot be omitted.

*E. Bodewig* (The Hague).

Oldenburger, Rufus. *Practical computational methods in the solution of equations*. Amer. Math. Monthly 55, 335-342 (1948).

The author gives some hints for finding an approximate value of the numerically largest real root of an algebraic equation; further, for finding by trial an approximate value of a quadratic function  $x^2+ex+f$  which has two conjugate complex roots.

*E. Bodewig* (The Hague).

Rossier, Paul. *Sur la multisection de l'angle et la trisection de Lambert*. Arch. Sci. Soc. Phys. Hist. Nat. Genève 1, 383-384 (1948).

Proško, V. M. *An electrical apparatus for the solution of systems of compatible linear algebraic equations*. Trav. Inst. Math. Stekloff 20, 117-128 (1947). (Russian)

The apparatus solves ten linear equations in ten unknowns. It employs the Gauss-Seidel iterative method in the same way that it is used by C. E. Berry et al. [J. Appl. Phys. 17, 262-272 (1946); these Rev. 7, 488] and the circuit involved is in many ways similar to theirs. The convergence of the process is not discussed; the equations are assumed to satisfy the conditions given by R. von Mises and H. Pollaczek-Geiringer [Z. Angew. Math. Mech. 9, 58-77, 152-164 (1929)].

Each equation  $\sum a_{ij}x_j + p_i = 0$  is realized in turn by producing the products  $a_{ij}x_j$  as direct current voltage drops across a set of eleven potentiometers:  $R_1, \dots, R_{10}$  for the unknowns  $x_j$  and  $R_0$  for the scale factor  $\epsilon$  associated with  $p_i$ . Each of these potentiometers can be shunted across a corresponding wire wound coefficient potentiometer  $r_j$ . The individual terms  $a_{ij}x_j$  and  $p_i$  are added by applying the corresponding voltages to the eleven independent coils of a specially constructed galvanometer. Instead of having a potentiometer  $r_j$  for each of the 110 coefficients, as in the machine described by Berry et al., there is only a sliding contact for each coefficient. One of these can be positioned on each of ten rods equally spaced around an axis so as to form a cylinder. There are eleven of these cylinders. By means of a handwheel and flexible wire drive they can be rotated in unison so that the sliding contacts can be brought in turn under eleven long narrow windows with linear scales for introducing the coefficients. When the handwheel is in the  $i$ th position the  $i$ th sliding contact from each cylinder is in contact with a corresponding linear resistance  $r_j$ , placed parallel to the axis of the cylinder, thus connecting resistances proportional to the coefficients of the  $i$ th equation across the corresponding resistance  $R_j$ . The position of the contact on  $R_j$  corresponds to  $x_j$ . It and the center tap of  $R_j$  are connected to the  $j$ th galvanometer coil. The  $r_j$  are also center-tapped to provide for negative values of the  $a_{ij}$ .

The apparatus represents the embodiment of design features practically all of which were discussed in detail by

the author [Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] (N.S.) 3, no. 4, 195-206 (1939)] when he undertook to improve on an experimental three-equation model using the same circuit (with a separate resistance, however, for each coefficient) which was built and described by K. V. Samsanov [Appl. Math. Mech. [Prikl. Mat. Mech.] (1) 2, 309-313 (1935)]. Here he explained why the multi-coil galvanometer used by Samsanov was preferred to other means for addition. Here also is the detailed discussion of errors on the basis of which the present apparatus is designed. In particular, the currents carried by the resistances and their magnitudes relative to the resistance of the galvanometer coils and its sensitivity are determined so that the error arising from nonlinear variation of the various resistances with the position of the contacts, because of the parallel resistances, remains within a previously set 0.5% limit. The requirement that coefficients be set in and unknowns read with less than 0.5% error determines the length of all resistances as 40 cm. A subsequent paper is to discuss actual performance and the possibility of automatic operation.

R. Church (Annapolis, Md.).

Cooper, J. L. B. The solution of natural frequency equations by relaxation methods. Quart. Appl. Math. 6, 179-183 (1948).

Treating the equations  $\sum a_{rs}x_s = \lambda \sum b_{rs}x_s$ , where  $(a_{rs})$ ,  $(b_{rs})$  are positive matrices and the modes  $x_s^{(0)}$  are normalised, the author establishes the following theorem which simplifies the solution of the equations and is simpler than that of T. S. Wilson. The system  $\sum a_{rs}^1x_s = \mu \sum b_{rs}x_s$ , where  $a_{rs}^1 = a_{rs} - a_{rs}^{(0)}$ ,  $a_{rs}^{(0)} = \sum_k a_{rk}x_k^{(0)} \cdot \sum_k a_{sk}x_k^{(0)}$ , has the same modes as the original system and the same proper values, with the exception of  $\mu^{(0)}$ , which has become zero. Furthermore, the author gives the following criterion for the convergence of the usual relaxation process. It converges to the lowest (highest) mode if the initial values  $x$  are so chosen that the initial value  $\lambda_1$  is less (larger) than  $a_{kk}/b_{kk}$  for all  $k$ .

E. Bodewig (The Hague).

Kerkhofs, W. Résolution de systèmes d'équations simultanées à un grand nombre d'inconnues. Ossature Métallique 12, 187-195 (1947).

Starting from statical considerations the author had developed an iteration method for solving linear equations. In this paper he completes the method by enumerating some subcases.

E. Bodewig (The Hague).

Obrechkoff, N. Sur les quadratures mécaniques. Spissi Bulg. Akad. Nauk 65, 191-289 (1942). (Bulgarian. French summary)

Making use of Cesàro's arithmetic means, the author develops a general integration formula, depending on two parameters  $n$  and  $k$ , from which numerous special formulae are derived. One of them is

$$J = \int_0^b f(x)dx = \frac{1}{2}b(f_0 + f_b) + \sum_{m=2}^n \frac{\binom{n}{m}}{m!} [f_0^{(m-1)} + (-1)^{m-1} f_b^{(m-1)}] + R,$$

$$R = \frac{(-1)^n}{2n+1} \left\{ \frac{n!}{(2n)!} \right\}^2 b^{2n+1} f^{(2n)}(\theta), \quad 0 < \theta < b.$$

It gives a better approximation than the similar formula of

Euler-Maclaurin. Another useful formula is

$$J = \frac{1}{2}b(f_0 + 4f_1 + f_b) + \frac{1}{120}b^3(f_0' - f_b') + \frac{1}{120}b^3f_m'' + R,$$

where  $m = b/2$ ,  $|R| \leq b^7 M/4838400$ ,  $M = \max |f^{(i)}(x)|$  for  $0 \leq x \leq b$ . In addition, the author gives a generalization of Newton's formula and inequalities useful for the formulae of Poncelet and Parmantier.

E. Bodewig (The Hague).

Chakrabarti, M. C. Remainders in quadrature formulae.

Bull. Calcutta Math. Soc. 39, 119-126 (1947).

Let the quadrature formula be

$$\int_0^b f(x)dx = \sum_{i=1}^p R_i f(x_i) + L(f)$$

with  $0 \leq x_1 < \dots < x_n \leq b$ . Let the remainder have the form  $L(f) = \sum A_r f(x_r) + \sum B_r f'(x_r) + \sum C_r f''(x_r) + \dots$  and let  $L(x^q) = 0$  for  $q = 0, 1, \dots, n-1$ , but  $L(x^n) = E \neq 0$ . Then the quadrature formula ("of order  $n$ ") is said to be "simplex" if, for every function  $f(x)$  having derivatives up to the  $n$ th order,  $L(f) = 0$  implies  $f^{(n)}(X) = 0$ , where  $X$  lies somewhere in the interval of the  $x_i$ . For such simplex formulae it is true that  $L(f) = (E/n!) f^{(n)}(X)$ . The author gives two old and two new theorems for deciding if a formula is simplex.

E. Bodewig (The Hague).

Kirkby, S. The relative accuracy of quadrature formulae of the Cotes' closed type. Coll. Aeronaut. Cranfield. Rep. no. 17, 6 pp. (1948).

The author makes a study of the error involved in the use of the Newton-Cotes numerical integration formula. It is shown by a consideration of the leading term in the error series that the accuracy of the integration resulting from  $2n+1$  applications of a  $2n$ -strip closed type formula is always better than that resulting from  $2n$  applications of a  $(2n+1)$ -strip closed type formula. For example, the relative error of three applications of Simpson's one-third rule (2-strip formula) as against two applications of Simpson's three-eighths rule (3-strip formula) is 0.444. Similarly, the relative error for the 4-strip formula compared with the 5-strip is 0.465, and for the 6-strip formula compared with the 7-strip is 0.475.

S. Levy (Washington, D. C.).

Aprile, Giuseppe. Un integrafo per la valutazione delle espressioni simboliche del calcolo operatorio funzionale. Pont. Acad. Sci. Comment. 8, 31-34 (1944).

The evaluation of the integral  $W(t) = \int_0^t G(t-\tau) V(\tau) d\tau$  is mechanized by displacing the graphs of the functions  $G(\tau)$  and  $V(\tau)$  by a distance  $t$ , displacing the wheel of an integrator by a distance  $G(t-\tau)$  from the center of the disc which is driven by the variable  $\tau$  over the interval zero to  $t$ , displacing the wheel of a second integrator by the distance  $V(\tau)$  and rotating the disc of the second integrator by the output from the wheel of the first integrator. The output of the second integrator is the desired value. Similar techniques have been used in differential analyzers.

M. Goldberg (Washington, D. C.).

\*Crank, J. The Differential Analyser. Longmans, Green & Co., London, New York, Toronto, 1947. viii+137 pp. (4 plates). \$2.50.

The construction and operation of a typical differential analyzer is described in considerable detail. A separate chapter is devoted to the construction of a small-scale machine using Meccano parts. Various techniques used in putting problems on the machine are described, and a

number of illustrations of actual problems are given. There is a brief treatment of the use of the differential analyzer on certain types of partial differential equations.

In view of the effort put into this book, the final result is disappointing. The author seems to be so preoccupied with a great mass of details that it is difficult for the reader to grasp the basic principles involved. Furthermore, much of the detail given is applicable to one particular machine only. A notable feature of the book is the extensive bibliography included at the ends of chapters.

S. H. Caldwell (Cambridge, Mass.).

Weibel, E. E., Cokyucel, N. M., and Blau, R. E. A mechanical analyzer for the solution of vibration problems of a single degree of freedom. *J. Appl. Mech.* 15, 146-150 (1948).

A torsional pendulum is used as the basic analogue device. Linear and nonlinear elastic members and various damping devices are provided. Forcing functions can be introduced graphically to meet any specification. Results are recorded photographically.

S. H. Caldwell (Cambridge, Mass.).

Tea, Peter L. A mechanical integrator for the numerical solution of integral equations. *J. Franklin Inst.* 245, 403-419 (1948).

The magnitude and scope of modern developments in large-scale computing machines may at times create a false scale of values. In contrast to these large, general-purpose machines, the device described in this paper might be regarded as just another special-purpose integrator. No claims are made for high precision, extraordinary speed, or even ease of operation (at least one operator would have to be rather dexterous in many cases). But the author has described a very simple mechanism, which can be built at low cost and which, in the hands of a patient user, can produce solutions with reasonable accuracy of the difficult problems expressed by various types of integral equations.

S. H. Caldwell (Cambridge, Mass.).

Lyusternik, L. A. Remarks on the numerical solution of boundary problems for Laplace's equation and the calculation of characteristic values by the method of networks. *Trav. Inst. Math. Stekloff* 20, 49-64 (1947). (Russian)

The method of successive approximations as used in solving the vector equation (1)  $x = Ax + y$ , where  $y$  is known,  $x$  is an  $n$ -dimensional vector and  $A$  is an  $n$ -dimensional matrix, consists in starting with an arbitrary vector  $x^0$  and forming successively the system of vectors  $x^1, x^2, \dots$ , where (2)  $x^k = Ax^{k-1} + y$ . If the sequence  $x^k$  converges to the vector  $x$ , it is said to be the solution of equation (1). However, the convergence may be very slow, in which case the problem is to extrapolate the approximations  $x^1, \dots, x^k$  already found so as to make the convergence more rapid without directly substituting in equation (2).

The author first considers the simple case when  $A$  is a symmetric matrix. Letting  $\lambda_1, \dots, \lambda_k$  be the characteristic values of  $A$ , and  $u_1, \dots, u_k$  the orthonormal system of the characteristic vectors, it follows that  $x^k - x = \sum \lambda_i^k c_i u_i$ . Thus, in particular, when a single greatest characteristic value  $\lambda$  exists,

$$x^k - x \sim \frac{1}{1-\lambda} (x^k - x^{k+1}) \sim \frac{1}{1-\lambda^k} (x^k - x^{k+2}),$$

and therefore  $x$  can be found when  $x^k, x^{k+1}, x^{k+2}$  are known. The case when two greatest (in absolute value)  $\lambda$ 's exist is also considered.

As an illustration the method is applied to the numerical calculation of the solution of plane boundary problems of Laplace's equation by the method of networks. Here the problem reduces to finding a function  $u$  which satisfies, in addition to certain boundary conditions, the equation  $u = Du$ , where

$$\begin{aligned} Du &= D^k u = \frac{1}{h^2} \nabla^k u + u, \\ \nabla u &= \partial^2 u / \partial x^2 + \partial^2 u / \partial y^2, \\ \nabla^k u &= 1/h^2 [u(x+h, y) + u(x-h, y) + u(x, y+h) \\ &\quad + u(x, y-h) - 4u(x, y)]. \end{aligned}$$

A numerical example is given.

The method is also applied to the Sturm-Liouville equation. S. D. Zeldin (Cambridge, Mass.).

## MECHANICS

Simon, Herbert A. The axioms of Newtonian mechanics. *Philos. Mag.* (7) 38, 888-905 (1947).

L'auteur discute l'exposé de Mach qui apparaît évidemment aujourd'hui obscur et insuffisant. Il cherche à approfondir le choix des masses, de l'horloge et du système de référence. Le mouvement dans l'intervalle ( $a \leq t \leq b$ ) par rapport à des axes d'origine  $O$  d'un système de  $n$  points est dit isolé s'il existe des constantes  $m_i \neq 0$  telles que

$$\sum m_i \frac{d^2}{dt^2} \vec{OP}_i = 0, \quad \sum m_i \frac{d^2}{dt^2} \vec{OP}_i \wedge \vec{OP}_i = 0.$$

Tel est l'exemple des mouvements disjoint, c'est-à-dire dont les accélérations des  $P_i$  sont toutes nulles. L'auteur étudie les mouvements isolés et le cas d'unicité des  $m_i$  à un facteur près. Il montre qu'étant donné un mouvement, on peut en général par changement d'axes en déduire un mouvement isolé avec des  $m_i$  choisis à l'avance. Il cherche aussi à obtenir un mouvement disjoint par une transformation sur le temps seul ou sur le temps et l'espace. Il recherche enfin

dans les systèmes partiels d'un mouvement isolé, les forces intérieures. Malgré l'intérêt de ces problèmes, ils ne me semblent pas très utiles pour une axiomatique générale de la mécanique, qui peut se réduire à très peu de chose si l'on veut vraiment séparer mathématiques et physique, et l'auteur signale d'ailleurs qu'il y aurait des difficultés à étudier à son point de vue des distributions plus générales de masses [introduites en toute généralité dans mon ouvrage "Les Principes Mathématiques de la Mécanique Classique," Arthaud, Grenoble-Paris, 1945; ces Rev. 7, 223].

M. Brelot (Grenoble).

Giorgi, Giovanni. I postulati della statica. *Pont. Acad. Sci. Comment.* 7, 531-543 (1943).

Staring, A. J. On central motions, in particular that along an ellipse. *Simon Stevin* 25, 208-223 (1947). (Dutch)

The author gives a general formula for the acceleration of a central motion, including the case that the center is at infinity. He considers in particular central motions along a

conic and deduces the law of force when the orbit is an ellipse. An apparatus is described for the demonstration of the motion of a planet and of the components of a double star.

O. Bottema (Delft).

**Janković, Z.** Cycloid as a tautochrone and brachistochrone. *Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II.* 2, 49-72 (1947). (Croatian. English summary)

**de Kok, F.** On the determination of small oscillations with two degrees of freedom. *Simon Stevin* 25, 228-230 (1947). (Dutch)

The author determines the vibrations about equilibrium of a system with two degrees of freedom without using the theory of quadratic forms. *J. Haantjes* (Amsterdam).

**Hoppmann, W. H., 2nd.** Impact of a mass on a damped elastically supported beam. *J. Appl. Mech.* 15, 125-136 (1948).

The motion of a beam and a mass subsequent to impact is studied. The force distribution (in time) is assumed to be approximately that given by the Hertz theory, and both damping and an elastic foundation for the beam are considered. An expression is found for the coefficient of restitution and for the energy associated with the various beam vibration modes. Two numerical examples are worked out in detail and discussed. *G. F. Carrier.*

**Lampariello, Giovanni.** Sur la dynamique du point matériel de masse variable. *C. R. Acad. Sci. Paris* 227, 35-37 (1948).

The author makes an application to rocket motion of the laws of motion as enunciated by Levi-Civita for a material particle of changing mass. He proposes to consider the case in which the mass of the rocket is a linear function of the time, using Picard's method of iteration. A first approximation is proposed as a power series in  $t$  (three terms being given). The author remarks that it is not difficult to calculate successively improved approximations as series in powers of the time. *A. A. Bennett* (Providence, R. I.).

**Batschelet, Ed.** Über einen Ausnahmefall des Wiedergehauersatzes von Poincaré. *Experientia* 4, 270 (1948).

An elementary proof of the well-known fact that, in the billiard ball problem on an elliptical table, a motion through a focus is asymptotic in both senses to the major axis.

*G. A. Hedlund* (New Haven, Conn.).

### Hydrodynamics, Aerodynamics, Acoustics

**Taub, A. H.** Relativistic Rankine-Hugoniot equations.

*Physical Rev.* (2) 74, 328-334 (1948).

By considering the fluid as a collection of a number of particles with rest mass  $m$ , the author first shows that, on the basis of kinetic theory, the internal energy per unit rest mass  $\epsilon$ , measured by an observer at rest with respect to the element of the fluid, must satisfy the inequality

$$\epsilon \geq \frac{3}{2} \frac{p}{\rho^0} + c^2 \left[ \left\{ 1 + \frac{9}{4} \left( \frac{p}{\rho^0 c^2} \right)^2 \right\}^{\frac{1}{2}} - 1 \right].$$

where  $p$  is the pressure and  $\rho^0$  is the rest density. In par-

ticular, if  $\epsilon = (\gamma - 1)^{-1} p / \rho^0$ , then  $\gamma < \frac{5}{3}$ . This restriction removes the difficulty of having a velocity of sound greater than the velocity of light in a vacuum. The author then considers the one-dimensional motion of a perfect gas, with a discussion on progressive waves. The paper concludes with a derivation of the Rankine-Hugoniot equations and the determination of the shock velocity, which is shown to be always less than that of light in a vacuum.

*H. S. Tsien* (Cambridge, Mass.).

**Cărstoian, I.** Sur un mouvement fluide de Beltrami. *Acad. Roum. Bull. Sect. Sci.* 28, 270-272 (1946).

A Beltrami motion is defined to be a fluid motion in which streamlines coincide with vortex lines. Appell has shown that if in a Beltrami motion  $\lambda$  is the ratio of the vorticity to fluid speed, then  $\lambda$  satisfies everywhere in the fluid the relation  $2\lambda = \vec{z} \cdot \vec{\omega}$ , where  $\vec{z}$  is the unit vector along the velocity vector [Appell, *Traité de Mécanique Rationnelle*, 3d ed., Gauthier-Villars, Paris, 1921, v. 3, pp. 414-415]. The author shows that the converse is also true and gives an integral form for the Appell condition.

*W. J. Nemerever* (Ann Arbor, Mich.).

**Cărstoian, I.** Sur la possibilité des mouvements tourbillonnaires à  $\Omega = \text{const.}$  d'un fluide parfait incompressible. *Acad. Roum. Bull. Sect. Sci.* 28, 503-504 (1946).

In this note it is shown that the motion of a perfect, incompressible fluid, subjected to conservative forces, with constant vorticity, is possible only when the deformation quadric is a hyperboloid. In that case the vortex vector is located along a generator of the cone asymptotic to the deformation quadric.

*W. J. Nemerever*.

**Cărstoian, I.** Sur le mouvement tourbillonnaire à  $\Omega = \text{const.}$  d'un fluide parfait incompressible. *Acad. Roum. Bull. Sect. Sci.* 28, 589-592 (1946).

This note studies a simple example of constant vorticity motion of a perfect incompressible fluid, namely, plane motion. The author derives the following geometric properties of the deformation quadric (a cylinder): (1) during the motion, vortex line length is preserved and the vortex surface is carried into a vector surface, and (2) the initial vortex surface is applicable to the vortex surface at any subsequent time. *W. J. Nemerever* (Ann Arbor, Mich.).

**Cărstoian, I.** Sur le vecteur tourbillon de l'accélération et les fonctions qui s'y rattachent. *Acad. Roum. Bull. Sect. Sci.* 29, 207-214 (1946).

From the properties of the curl of the acceleration vector of a perfect fluid several results from classical hydrodynamics are established in a new way. Among these are: (1) the existence of a velocity potential implies the existence of an acceleration potential [see E. Vessiot, *Bull. Sci. Math.* (2) 35, 233-244 (1911)], (2) the existence of a 2-dimensional acceleration potential implies the invariance with respect to time of the vortex strength [see L. Lichtenstein, *Grundlagen der Hydromechanik*, Springer, Berlin, 1929, p. 409], and (3) the divergence of the acceleration of an irrotational incompressible fluid is essentially positive for non-translatory motion [see C. Jacob, *Bull. Math. Soc. Roumaine Sci.* 46, 81-90 (1944); these Rev. 8, 106]. [Numerous typographical errors are in evidence; note especially equation 13 where  $\partial \xi / \partial t$  should be  $d\xi / dt$ .] *W. J. Nemerever*.

**Putman, Henri J. Unsteady flow in open channels.** Trans. Amer. Geophys. Union 29, 227-232 (1948).

The paper is a résumé of some of the work done by J. Massau [Annales de l'Association des Ingénieurs Sortis des Ecoles Spéciales de Gand (1) 12, 185-444 (1889); (2) 23, 95-214 (1900)] on the problems of unsteady flow of water in open channels. It seems likely that Massau was the first to notice that the partial differential equations for these phenomena can be integrated by Riemann's method; Massau also carried out the solutions numerically in a variety of cases of interest in hydraulics by using what is essentially the method of finite differences applied to the differential equations written in characteristic form. Among the cases presented by the author is, for example, the problem of the development of a bore (analogous to a shock in gas dynamics) from a continuous motion, and this illustrates the pioneering nature of Massau's work. *J. J. Stoker.*

**Fromm, Hans. Laminare Strömung Newtonscher und Maxwellscher Flüssigkeiten.** Z. Angew. Math. Mech. 25/27, 146-150 (1947).

The author points out that the time rate of the stress tensor which appears in the stress-strain law of Maxwell is the material rate which can be represented as the sum of the local and convective rates and a third rate due to the rotation of the volume element. The resulting theory is applied to the laminar flows between parallel plates and in a tube of circular cross section. *W. Prager.*

**Cocchi, Giovanni. Sull'equazione generale del moto nelle correnti liquide.** Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) 3, 207-213 (1947).

**Krall, Giulio. Sul calcolo del rollio di un galleggiante tenendo conto dell'inerzia del fluido.** Pont. Acad. Sci. Acta 8, 107-117 (1944).

**Polubarinova-Kočina, P. Ya., and Fal'kovič, S. V. The theory of seepage of a fluid in porous media.** Akad. Nauk SSSR. Prikl. Mat. Meh. 11, 629-674 (1947). (Russian)

This is a comprehensive report on the Russian contributions to the theory of seepage of an incompressible fluid through a porous medium. It gives a review of a great number of exact solutions for the corresponding steady and unsteady motion with or without a free surface. The paper contains a seven page bibliography. *A. Weinstein.*

**Faxén, O. H. Forces exerted on a rigid cylinder in a viscous fluid between two parallel fixed planes.** Acta Polytech., no. 2, 1-13 (1947).

[Also issued as Ingenjörsvetenskapsakademiens Handlingar, no. 187 (1946).] This paper deals with the problem of resistance exerted on a unit section of an infinitely long circular cylinder which moves normal to its axis with a constant velocity  $u_0$ , in the direction of  $x$ , midway between two fixed parallel planes, in a viscous incompressible fluid. The mathematical problem is simplified by assuming that (i) the motion is slow so that the inertia force is unimportant and (ii) the diameter of the cylinder is small in comparison with the distance between the parallel planes. The author then poses the following problem: to solve the two-dimensional Stokes equations subject to the boundary conditions (a) the velocity components  $u$  and  $v$  vanish on the parallel planes and (b)  $u = u_0$  and  $v = 0$  on the moving cylinder. The problem as formulated appears to be an unsteady one and,

presumably, the time variation is relatively insignificant and therefore neglected. Both resistance and pressure are evaluated numerically. *Y. H. Kuo* (Ithaca, N. Y.).

**Bouligand, Georges. Entrainement d'un liquide visqueux dans un vase annulaire.** C. R. Acad. Sci. Paris 226, 2106-2108 (1948).

**Moreau, Jean-Jacques. Sur l'allure à l'infini d'un écoulement permanent lent.** C. R. Acad. Sci. Paris 224, 1469-1472 (1947).

The author discusses the behavior of a steady motion of a viscous incompressible fluid which satisfies the Stokes equations, namely,  $\nu \Delta u_i + \partial p / \partial x_i = 0$  and  $\partial u_i / \partial x_i = 0$ ,  $u_1$ ,  $u_2$ ,  $u_3$  and  $p$  being, respectively, the velocity components and pressure at a point  $(x_1, x_2, x_3)$ . The author proves that if the velocity tends to zero at infinity at least as rapidly as the reciprocal of the distance from a fixed point, then the pressure tends to a constant  $p_0$ . *Y. H. Kuo.*

**Rivlin, R. S. The hydrodynamics of non-Newtonian fluids.** I. Proc. Roy. Soc. London. Ser. A. 193, 260-281 (1948).

The general structure of the relation between stress and velocity-strain in an incompressible viscous fluid is established, and two problems of steady laminar flow are discussed, viz., the torsional motion of a cylindrical mass of fluid, produced by forces applied to the plane end surfaces of this mass, and the torsional motion of a mass of fluid between infinite coaxial cylinders rotating with different angular velocities about their common axis. *W. Prager* (Providence, R. I.).

**Burgers, J. M. On the influence of gravity upon the expansion of a gas.** II. Nederl. Akad. Wetensch., Proc. 51, 525-532 (1948).

Continuing the study of the expansion of a vertical column of gas into a vacuum when the upper partition is suddenly removed [same Proc. 51, 145-154 (1948); these Rev. 9, 633], the author calculates the motion in the vicinity of the first shock by using power series expansions of appropriate variables. A first approximation to the trace of the shock is obtained by observing that in such an approximation the entropy change across the shock can be neglected and the analytic continuation of the solution before the shock is the solution after the shock. Then the condition of continuity of space and time at the shock gives a first approximation of the trace of the shock. Higher approximations are then obtained by a perturbation method. The state of the gas after the shock is then determined by the Rankine-Hugoniot conditions. Finally a method of analyzing the flow after the shock is indicated.

*H. S. Tsien* (Cambridge, Mass.).

**von Kármán, Theodore. Problems of flow in compressible fluids.** Ciencia y Técnica 111, 1-26 (1948). (Spanish. English summary)

Translation of University of Pennsylvania Bicentennial Conference, Fluid Mechanics and Statistical Methods in Engineering, pp. 15-39, University of Pennsylvania Press, Philadelphia, Pa., 1941; these Rev. 4, 59.

**Neményi, P., and Prim, R. Some geometric properties of plane gas flow.** J. Math. Physics 27, 130-135 (1948).

For steady plane rotational flow of a perfect gas subject to no external forces, it is shown that the streamlines are either concentric circles or parallel straight lines, if either

the velocity magnitude, or the vorticity, is constant along each streamline, or if the isolines ( $q=\text{constant}$ ) coincide with the isocurves ( $\omega=\text{constant}$ ). *M. H. Martin.*

**Kiselev, B. M. Calculation of one-dimensional gas flows.**

*Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.]* 11, 177-192 (1947). (Russian. English summary)

[The author's name is misprinted Kiseley in the title of the summary.] A one-dimensional analysis is provided of the steady flow of a gas through a channel. Most of the results obtained having a general character are well known (e.g., the discontinuities in state variables associated with a shock in a tube are given). The flow of a mixture of two gases through an ejector is considered in some detail.

*G. F. Carrier* (Providence, R. I.).

**Robinson, A. On source and vortex distributions in the linearised theory of steady supersonic flow.** *Coll. Aeronaut. Cranfield. Rep. no. 9, 27 pp. (1 plate)* (1947).

The concept of the finite part of an integral (as introduced by Hadamard) and the hyperbolic counterparts of the classical vector operators like div, grad, etc. are discussed for the linearized equation for stationary supersonic flow. With the aid of these concepts some of the technique familiar from the theory of incompressible fluids may be easily applied to supersonic flow. As examples, the author derives some fundamental properties of flow due to a distribution of supersonic sources, doublets or vortices. *P. A. Lagerstrom.*

**Tollmien, W. Steady two-dimensional and rotationally-symmetric supersonic flows.** The Graduate Division of Applied Mathematics, Brown University. Translation no. A9-T-1. v+152 pp. 1948.

[Translation of Technische Hochschule Dresden, Peene-munde Archiv 44/1-44/6 (1940).] In this paper the equations of inviscid gas flow, adiabatic but not necessarily isentropic, and of the stationary shock wave, are fully discussed. The general theory of characteristics of second-order quasi-linear equations in two variables, and of their use in the numerical solution of the equations, is set out by choosing curvilinear coordinates  $\xi, \eta$ , finding the condition that a curve  $\eta=\text{constant}$  satisfies one property of characteristics, and then deducing the other properties. It is applied in three chapters to plane and axisymmetrical potential flow and to plane flow with vorticity (due to the presence of nonuniform shock-waves). The first two of these are modifications of the Prandtl-Busemann method and of Guderley's account [Jahrbuch 1940 der Deutschen Luftfahrtforschung, 1522-1535 (1940)]. The third is original and will be compared with the unpublished (though fairly well known) work of various authors including Guderley and R. Meyer. *M. J. Lighthill* (Manchester).

**Isenberg, J. S. The method of characteristics in compressible flow. I. (Steady supersonic flow).** Prepared under the supervision of C. C. Lin. Tech. Rep. no. F-TR-1173A-ND (GDAM A-9-M II/1). Headquarters Air Materiel Command, Wright Field, Dayton, Ohio. xxiii+219 pp. (1947).

This is essentially a manual on the derivation, technique and application of the known procedures in the method of characteristics in steady supersonic flow. An introduction provides the background for the method of characteristics through the theory of the general system of first order

quasi-linear hyperbolic differential equations in two dependent and two independent variables. Chapter I derives the characteristic equations, i.e., the equations for the characteristic curves and the relations connecting the changes of the dependent variables along the characteristics, for plane and axially symmetric flows which may be either rotational, with or without constant stagnation enthalpy, or irrotational. The derivations are made for various combinations of state variables as dependent variables. Chapter II, which is the main portion of the work, is a very detailed and complete account of the numerical and graphical procedures for integrating the characteristic equations. The procedures are of four general types, depending on the dependent variables in the characteristic equations and on special circumstances of the flow, and can be adapted to both lattice point and mesh methods. Chapter III consists of applications to certain classes of flows, such as nozzle flows and flows with shock waves. Here the emphasis is on determining from the given data of the problem the initial values necessary for proceeding by the method of characteristics. Methods for calculating attached shock waves are shown in detail. There is an extensive bibliography. *D. Gilbarg.*

**Pekeris, C. L. Stability of the laminar flow through a straight pipe of circular cross-section to infinitesimal disturbances which are symmetrical about the axis of the pipe.** *Proc. Nat. Acad. Sci. U. S. A.* 34, 285-295 (1948).

The author shows that the laminar flow through a pipe of circular cross-section is stable with respect to axially symmetrical disturbances. Stability with respect to disturbances of the torsional type has been shown by Synge. "The stability to meridional perturbations was studied by Sexl, but his treatment is shown to be incomplete. In this investigation, an explicit expression is derived for the characteristic value in the case of a meridional disturbance which is valid for small Reynolds numbers. An asymptotic expression for the [characteristic value]  $C$  valid for large [Reynolds numbers]  $R$  is also derived." Further details are given, which make the investigation very thorough.

*C. C. Lin* (Cambridge, Mass.).

**\*Dryden, Hugh L. Recent advances in the mechanics of boundary layer flow.** *Advances in Applied Mechanics*, edited by Richard von Mises and Theodore von Kármán, pp. 1-40. Academic Press, Inc., New York, N. Y., 1948. \$6.80.

**Cope, W. F., and Hartree, D. R. The laminar boundary layer in compressible flow.** *Philos. Trans. Roy. Soc. London. Ser. A.* 241, 1-69 (1948).

This paper discusses the various methods of integrating the laminar boundary-layer equations for a compressible fluid. It is shown that when there is a pressure gradient, some of the usual methods for the incompressible case (such as the Pohlhausen method) become very complicated in the compressible case, and are probably rather inaccurate. Methods recommended include (i) integration by series expansions, (ii) step-by-step integration in one of the variables, and (iii) the method of finite differences for both independent variables. The first method is developed in detail, because even in the second method it is best to start the calculation by using some limited results obtained from series expansions. The series used are expansions in one independent variable with coefficients which are functions of the others. It is found that the independent variables can be so chosen that the differential equations for the coeffi-

cients in the expansions have the same general structure as for an incompressible fluid. The boundary conditions and the limiting forms of the equations for zero Mach number are investigated. The application of iterative methods to the equations is discussed. Finally, the methods of applying the ENIAC to obtain solutions of the above equations are described in some detail. Tables of results are given. Most of the work concerns the case of no heat transfer. Some discussion of the case with heat transfer is added at the end.

C. C. Lin (Cambridge, Mass.).

**Crocco, Luigi.** *Lo strato limite laminare nei gas.* Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. 187, 78 pp. (1947).

[Also issued as Monografie Scientifice di Aeronautica, no. 3 (1946).] The laminar boundary layer equations for a gas are transformed into two equations for  $\tau$ , the viscous stress, and  $\rho$ , the density; with  $x$ , the coordinate along the surface, and  $u$ , the  $x$ -component of the velocity, as independent variables. These equations are solved fairly simply when the main stream pressure is constant and either the Prandtl number is unity or the product of density and viscosity is constant. A recurrence method is given when neither of the last two assumptions holds. The second assumption is found to give much more accurate results for air than the first. Using it, the author is enabled to give a full description with many graphs of the laminar flow of gas past a flat plate. M. J. Lighthill (Manchester).

**Oswatitsch, K., and Wieghardt, K.** Theoretical analysis of stationary potential flows and boundary layers at high speed. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. 1189, 59 pp. (1948).

[Translation of Lilienthal Gesellschaft für Luftfahrtforschung, Ber. 513/1, pp. 7-24 (1942); the reference is incorrect in the translation under review.] The authors discuss the problem of two-dimensional transonic flow. The paper is divided into two parts. In the first part a discussion of potential flow is given and in the second part the boundary layer effects are taken up. The transonic potential equation  $\varphi_{xx} + (a^*/(\gamma+1))\varphi_{yy} = 0$  is developed and discussed in connection with the flow through the throat of a nozzle. A numerical method for the computation of flow fields is described. This method proceeds from a known linearized solution at large distances from the body and allows the computation of the supersonic zone. The body contour is obtained from the computation. Several examples are shown including cases where the maximum velocity does not occur at the maximum thickness point of the contour.

In the second part boundary layer effects are studied. Instability of the boundary layer in supersonic flow is shown to exist. The reason for the instability is the fact that the relation between pressure gradient and area change of a stream filament is different in subsonic as compared to supersonic motion. Thus the subsonic filaments in the boundary layer adjacent to the supersonic filaments in the free stream are in unstable equilibrium. Extensive computations of both laminar and turbulent boundary layers are given and the interaction of these boundary layers with a linearized outer supersonic stream are studied. Quantitative results showing the instability of the supersonic case and the stability of the subsonic case are presented. The authors conclude that the interaction between boundary layer and free stream is of paramount importance in transonic flow.

Reviewer's remark. It may appear that some of the results of this paper are well known and some obsolete. However, it should be noted that the original paper appeared in Germany in 1942; thus most of the investigations into this field in other countries were actually done later.

H. W. Liepmann (Pasadena, Calif.).

**Mangler, Werner.** Das Impulsverfahren zur näherungsweisen Berechnung der laminaren Reibungsschicht. Z. Angew. Math. Mech. 24, 251-256 (1944).

The author investigates possible extension of the range and accuracy of the Kármán-Pohlhausen method for calculating laminar boundary layer development by choosing a more general one-parameter family of profiles in the form

$$u/U = 1 - (1-\eta)^n(1+a_1\eta + a_2\eta^2 + \dots + a_n\eta^n), \quad \eta = y/\delta(x).$$

The values of the constants are related through the first of Prandtl's boundary layer equations and its first two derivatives with respect to  $y$  evaluated at the wall. The first two of these relations are algebraic, the third is a differential equation. Applying these conditions to the special cases  $r=1$  and  $r=2$  gives parameters  $\bar{\lambda} = \lambda/n(n+1)$  and  $\bar{\lambda} = 3\lambda/(n+1)(n+2)$ , respectively, where  $\lambda = U'\delta^2/\nu$  and the value of  $n$  is left undetermined. The Kármán integral equation reduces essentially to a first order differential equation in  $\bar{\lambda}$  and may be integrated either directly or numerically.

In order to estimate the accuracy of the approximation the author treats the case  $U/U_0 = k(x/L)^m$  which was evaluated numerically by Hartree; the flat plate ( $m=0$ ) and the stagnation point ( $m=1$ ) are considered in detail. The comparison shows that for particular values of  $n$ , depending upon whether the profile family for  $r=1$  or  $r=2$  is employed, the agreement with Hartree's results is very good but less satisfactory where a positive pressure gradient in the direction of flow is present. Further calculations for a circular cylinder and for an elliptic cylinder at 4 degrees angle of attack are presented; the former is compared favorably with the more accurate solution of Görtler.

The author deduces that his more complicated family of profiles give better agreement for negative pressure gradients and poorer agreement for positive pressure gradients than that of Pohlhausen. All such approximations appear to break down entirely for sufficiently strong positive pressure gradients. F. E. Marble (Pasadena, Calif.).

**Tetervin, Neal.** Laminar flow of a slightly viscous incompressible fluid that issues from a slit and passes over a flat plate. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 1644, 40 pp. (1948).

This report concerns the laminar flow of a viscous incompressible fluid that issues from a slit and passes over a flat plate. By the boundary layer approximations together with the similarity hypothesis, the author transforms, in a general manner, the dynamical equations into a nonlinear total differential equation of the third order. Let the origin of the Cartesian system  $(x, y)$  coincide with the slit and let  $y$  be normal to the plate. The velocity components  $u$  and  $v$  parallel, respectively, to  $x$  and  $y$  are of the form:

$$u = x^{-(2k+1)/2} f(\eta), \\ v = \frac{1}{2} x^{-(k+2)/2} \left[ (k+2) \int_0^y \frac{df}{d\eta} d\eta + (2k+1) \int_0^y f d\eta \right].$$

with  $\eta = yx^{-(k+2)/2}$ ,  $k = R^{-1} (df/d\eta)_0$ , where the Reynolds number  $R$  can be arbitrarily chosen. In the present problem,  $k$  was found to be  $\frac{1}{2}$  and the differential equation for  $\int_0^y f d\eta$  can then be solved numerically. Both the function

and its derivatives are tabulated. It was found that at the outer edge of the jet there is a normal flow toward the plate. When the plate is removed, i.e.,  $k=0$ , the problem reduces to that of a plane jet considered by Bickley [Philos. Mag. (7) 23, 727-731 (1937)].

Y. H. Kuo.

**Ray, M.** Flow of a liquid from a reservoir over a plane-boundary-layer theory. Philos. Mag. (7) 39, 409-412 (1948).

The boundary layer equations of Prandtl including the time variation are considered. The problem is reduced to that of a nonlinear ordinary differential equation by introducing the variables  $\psi = \sqrt{f(\eta)}xt^{-\frac{1}{2}}$ , where  $\eta = y(\nu t)^{-\frac{1}{2}}$ , obtained through the usual similarity considerations. The differential equation is

$$f'''(\eta) + [f(\eta) + \frac{1}{2}\eta]f''(\eta) - (f'(\eta))^2 + f'(\eta) = 0.$$

The appropriate solution is found by considering a series of positive powers of  $\eta$ , beginning with the quadratic term. The author finds a satisfactory approximation by restricting himself to four terms of the series, the coefficients of which are easily obtained. The author interprets the solution physically in terms of the nonsteady flow of a viscous liquid from a reservoir over a semi-infinite flat plate.

F. E. Marble (Pasadena, Calif.).

**Alden, Henry Leonard.** Second approximation to the laminar boundary layer flow over a flat plate. J. Math. Physics 27, 91-104 (1948).

The author considers the flow of a viscous fluid about a semi-infinite flat plate parallel to the direction of flow and calculates the first two terms of an asymptotic solution of the Navier-Stokes equations. Using parabolic coordinates the stream function is expanded in powers of the kinematic viscosity. The partial differential equations describing each of the first two coefficients in this expansion are reduced to ordinary differential equations through similarity considerations. The first of these equations together with its boundary conditions is formally identical with that obtained in the Blasius solution and the available numerical integrations apply. However, because the present equation involves parabolic coordinates the boundary layer profiles represented by the first term of the asymptotic expansion are not identical with the Blasius profile. The second of the equations is solved by numerical integration in the range near the plate and by an approximate analytic procedure at large distances from the plate.

From the Navier-Stokes equations, the pressure coefficient induced by the boundary layer growth was calculated and was found to deviate only slightly from the pressure of the free stream except in the immediate vicinity of the leading edge. The total resistance of the plate could not be determined because of a nonintegrable singularity in the shear force at the nose of the plate. Comparison between first and second approximations indicate that the second approximation is beyond the limit of accuracy for  $R_s \geq 500$ , where  $R_s$  is the Reynolds number based on the distance from the leading edge of the plate. At  $R_s = 100$  the second approximation is of noticeable magnitude and becomes of increasing importance as the leading edge is approached.

F. E. Marble (Pasadena, Calif.).

**Thwaites, B.** An exact solution of the boundary-layer equations under particular conditions of porous surface suction. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. 2241 (9656), 6 pp. (1946).

**Thwaites, B.** On certain types of boundary-layer flow with continuous surface suction. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. 2243 (9629), 6 pp. (1946).

**Sherman, D. I.** On Prandtl's equation in the theory of a wing of finite span. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk 1948, 595-600 (1948). (Russian)

A method for the solution of Prandtl's equation is given for the special case when the chord  $b(x)$  of the wing's profile is of the form  $b(x) = (a^2 - x^2)^{\frac{1}{2}}/p(x)$ , where  $a$  is the half span of the wing, and  $p(x)$  is a rational function of the form  $\sum a_k x^{2k} / \sum b_k x^{2k}$ . The author states that his method has the advantage over other methods [I. N. Vekua, Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 9, 143-150 (1945); these Rev. 7, 207] in that it is directly applicable (without the use of conformal mapping) to the more general cases when the path of integration in Prandtl's equation is not a straight line, and when the coefficients  $a_k$  and  $b_k$  in  $p(x)$  are not real. Such forms of Prandtl's equation occur in the theory of elasticity. The author also indicates a method for the approximate solution of Prandtl's equation for the case when  $b(x)$  is an arbitrary function. He states that it seems to him that this approximate solution can yield sufficiently accurate results in some cases for practical purposes.

H. P. Thielman (Ames, Iowa).

**Kaplan, Carl.** The flow of a compressible fluid past a circular arc profile. Tech. Rep. Nat. Adv. Comm. Aeronaut., no. 794, 26 pp. (1944).

The subsonic flow over a circular arc profile is analysed by the perturbation method developed by the author in connection with the flow over a thin symmetrical body [same Rep., no. 768 (1943); these Rev. 9, 477]. The circulation over the airfoil is determined by the Kutta-Joukowski condition that the velocity at the trailing edge, and hence also at the leading edge, is finite. There is then no stagnation point in the flow. Terms up to the third power in the camber ratio are calculated. The main results are as follows. (1) Up to the appearance of local sonic velocity, the relation between the ratio of velocity at any point over the surface to the free stream velocity and the free stream Mach number shows very little difference from the previous values for the symmetrical body. Thus the concept of velocity and pressure correction formulae for subsonic flows is justified. (2) For subsonic flows, the Kármán-Tsien formula gives results that are in close agreement with the author's more accurate calculations.

H. S. Tsiem (Cambridge, Mass.).

**Lighthill, M. J.** Supersonic flow past slender bodies of revolution the slope of whose meridian section is discontinuous. Quart. J. Mech. Appl. Math. 1, 90-102 (1948).

The linearized theory of slender bodies of revolution in supersonic flow, either symmetrical or yawed, is ordinarily based on the assumption that the meridian section is a continuous curve with continuous slopes [von Kármán and Moore, Trans. A.S.M.E. 54, 303-310 (1932)]. This is an attempt to extend the theory to shapes having discontinuous slopes. Instead of  $f(x) = -S'(x)/2\pi$ , where  $S(x)$  is  $\pi R^2(x)$ , the cross-sectional area distribution, the author uses for the source strength distribution  $f(x)$  the function

$$f(x) = -(2\pi)^{-1} \int_{a_0}^x S''(y) dy - \sum_{i=0}^n f_i(x),$$

where  $x = a_0$  denotes the nose, and the integral is a Lebesgue integral which ignores the points  $y = a_i$  where the integrand is not defined. Each  $f_i(x)$  is zero for  $x \leq c_i$  and continuous for  $x \geq c_i$ , where  $c_i = a_i - \alpha R(a_i)$ ,  $\alpha$  being the cotangent of the Mach angle. The author then deduces that there are extra terms, arising from the discontinuities, in the expressions for surface pressure and drag, that each discontinuity affects markedly the pressures in the region just behind it, and that these new contributions to the drag coefficient decrease as the Mach number increases. (If there are no discontinuities, the drag coefficient is independent of Mach number.) As an example, a double-cone projectile is considered, and numerical results are given. A similar investigation is made for the yawed projectile. Although the author again finds contributions to the pressure distribution due to the discontinuities, they integrate out in the lift and moment.

W. R. Sears (Ithaca, N. Y.).

Lighthill, M. J. Supersonic flow past slender pointed bodies of revolution at yaw. *Quart. J. Mech. Appl. Math.* 1, 76-89 (1948).

For a slender projectile at a small angle of attack  $\psi$ , the flow is expected to be nearly irrotational. In cylindrical coordinates  $r, \theta, x$ , the velocity potential is written

$$\phi = Ux \cos \psi + U \sin \psi r \cos \theta + \phi_0(x, r) + \psi \phi_1(x, r) \cos \theta + \psi^2 \phi_2(x, r, \theta) + \psi^3 \phi_3(x, r, \theta) + \dots$$

The first two terms represent the uniform stream. The equation of motion,  $2a^2 \nabla^2 \phi = \nabla q^2 \cdot \nabla \phi$ , is also written out in powers of  $\psi$ , leading to a succession of partial differential equations for  $\phi_0, \phi_1, \dots$ ; this process is carried only through  $\phi_3$ . The boundary conditions are approximated in the same manner. Expressions correct to order  $\psi^3$  are then worked out for the pressure coefficient  $C_p$ , the normal force  $L_1$ , the drag  $D$ , the lift  $L$  and the pitching moment  $M$ .

The author writes  $O(t^n)$  to mean  $O(t^m \ln^k t^{-1})$  for some  $k$  when  $r = O(t)$ . He solves the differential equations approximately, obtaining solutions for  $\phi_0, \phi_1, \phi_2, \phi_3$  that are correct to  $O(t^4), O(t^6), O(t^8), O(t^{10})$ , respectively. This leads to a formula for  $C_p$ , in which  $O(\psi^4), O(\psi^6), O(\psi^8), O(\psi^{10})$  are neglected. He calculates the drag, lift, and moment using this formula for  $C_p$ ; the lift coefficient is  $2\psi$  plus a small term; the drag due to yaw is proportional to  $\psi^2$  for flat-based projectiles and is zero for bodies pointed at both ends.

The author then returns to the differential equation for  $\phi_1$  to obtain a second approximation correct to  $O(t^4)$ . This permits calculation of the forces to higher accuracy; e.g., the lift now includes  $O(\psi^4)$ , etc. As an example, he computes the  $C_L$  and  $C_M$  slopes at  $\psi = 0$  for various Mach numbers between 1.5 and 3.0. Comparison is also made with the linearized theory of Tsien [J. Aeronaut. Sci. 5, 480-483 (1938)]; the results show considerable differences. The author believes that the new formulas are also correct (to the orders of approximation indicated) when the presence of a bow shock and subsequent rotational flow are considered.

W. R. Sears (Ithaca, N. Y.).

Garrick, I. E. On the plane potential flow past a lattice of arbitrary airfoils. *Tech. Rep. Nat. Adv. Comm. Aeronaut.*, no. 788, 16 pp. (1944).

Theodorsen's method [same Rep., no. 411 (1931)] of finding the potential flow about an arbitrary aerofoil in a uniform stream is extended to deal with the case of a cascade of congruent aerofoils. M. J. Lighthill (Manchester).

Krasil'shikova, E. A. The influence of the edges of the tips on the motion of a wing with supersonic velocity. *Doklady Akad. Nauk SSSR (N.S.)* 58, 543-546 (1947). (Russian)

The author wishes to find the velocity potential associated with the oscillation of a thin deformable obstacle in a supersonic stream when, at portions of the leading edge, the velocity component normal to that edge is subsonic (i.e., it has "subsonic leading edge" portions). The conventional linearized theory is adopted, the trailing edge is a "supersonic edge," and the "subsonic leading edges" are allowed to interact. The potential is formulated with the Green's function and the essential problem (that of evaluating the potential gradient normal to the obstacle plane in a certain region) is reduced to an integral equation. This integral equation is inverted for the special case where the wing is rigid and not oscillating by successive Abel integral equation inversions (i.e., Evvard's method). The next review is concerned with a subsequent paper inverting the integral equations for the general case. G. F. Carrier.

Krasil'shikova, E. A. The influence of the edges of the tips on the motion of a vibrating wing with supersonic velocity. *Doklady Akad. Nauk SSSR (N.S.)* 58, 761-762 (1947). (Russian)

The integral equation occurring in the paper of the foregoing review, namely

$$\int_0^x \int_{x(\xi)}^y \frac{\theta(\xi, \eta) \cos [\lambda(y-\eta)^{1/2}(x-\xi)^{1/2}]}{(y-\eta)^{1/2}(x-\xi)^{1/2}} dy d\xi = f(x, y)$$

is to be inverted. The cosine function is expanded in a power series of its argument and  $\theta$  is represented as  $\sum \theta_{2n}(x, y) \lambda^{2n}$ . It is finally possible to find  $\theta_{2n}$  as an  $n$ -tuple sum of terms containing  $2n$ -tuple integrals of  $f(x, y)$ . No examples are computed and no indication is given as to the modification in (say) dynamic lift implied by the results of these papers.

G. F. Carrier (Providence, R. I.).

Krasil'shikova, E. A. The influence of the vortex sheet on the stability of the motion of a wing with supersonic velocity. *Doklady Akad. Nauk SSSR (N.S.)* 58, 989-991 (1947). (Russian)

The work of the two papers reviewed above is modified to investigate the steady motion of a wing with "subsonic trailing edge" portions. At these edges, of course, the Jukowski condition must be applied. The problem again reduces to that of solving certain integral equations. When the trailing edges do not interact the integral equation is again inverted by Evvard's method. The inversion is not discussed for the more complicated case. Evvard has also solved this problem [see, e.g., Tech. Notes Nat. Adv. Comm. Aeronaut., no. 1585 (1948); these Rev. 9, 478].

G. F. Carrier (Providence, R. I.).

Ward, G. N. The approximate external and internal flow past a quasi-cylindrical tube moving at supersonic speeds. *Quart. J. Mech. Appl. Math.* 1, 225-245 (1948).

The linear theory is applied to calculate the external and internal flows about a nearly-cylindrical tube at small incidence. The potential equation (wave equation) is attacked by operational methods. The potential is calculated in four parts (external and internal, each in symmetrical flow and at incidence) and in each case the Laplace transform of the result appears as a combination of modified Bessel functions. These are amenable to interpretation by numerical integra-

tion, and also the initial and asymptotic behaviors are found by standard methods. The results obtained for the internal flow are restricted to a region of axial length  $2R(M^2-1)^{1/2}$  near the entrance, where  $R$  is the radius of the tube and  $M$  the stream Mach number. However, as the author points out, shock waves are expected actually to form within this region, so that further extension of the results downstream would be meaningless. The singularities that occur inside the tube, at zero incidence, are discussed in detail, even though the linearized theory is inapplicable to the physical problem. Such singularities result from a finite leading-edge angle or discontinuity of slope of the wall. Analogous phenomena in a nonlinear solution are said to be discussed by R. E. Meyer in a forthcoming paper. *W. R. Sears.*

**Browne, S. H., Friedman, L., and Hodes, I.** A wing-body problem in a supersonic conical flow. *J. Aeronaut. Sci.* 15, 443-452 (1948).

Several conical-flow situations representing wing-body combinations are treated. Both the symmetrical problem, relating to a wedge wing and conical body at no incidence, and the antisymmetrical, in which wing thickness is neglected and incidence considered, are attacked. In both cases, the leading edges are taken both outside and inside the Mach cone of the nose. The familiar transformation of the conical coordinates is employed, which reduces the linearized potential equation to Laplace's equation. The resulting formulas for the interaction potential are illustrated by means of numerical examples for all four cases. In the antisymmetric case, leading edges forward of Mach cone, the interaction is very small. *W. R. Sears* (Ithaca, N. Y.).

**Evvard, John C.** A linearized solution for time-dependent velocity potentials near three-dimensional wings at supersonic speeds. *Tech. Notes Nat. Adv. Comm. Aeronaut.*, no. 1699, 35 pp. (1948).

A source-distribution method is applied to derive a solution for the time-dependent surface velocity potential of thin finite wings at supersonic speeds.

*From the author's summary.*

**Ferrari, Carlo.** Interference between wing and body at supersonic speeds—theory and numerical application. *J. Aeronaut. Sci.* 15, 317-336 (1948).

The interference between wing and body at supersonic speed is calculated by using the linearized theory. The author assumes that the interference can be determined from the additional perturbation velocity potentials for the body and for the wing such that the normal disturbance velocity at the surface of the body and the wing due to the original perturbation velocity potentials of the isolated body and the isolated wing is cancelled. The wing chosen for study is of rectangular plan form and is without thickness and twist. For the calculation of the interference on the body by the wing, the author writes the additional perturbation potential  $\Phi^{(1)}$  as  $\Phi^{(1)} = V_0 \sum_m \Phi_m^{(1)}(x, r) r^m \sin m\theta$ , where  $V_0$  is the free stream velocity,  $x$  the axis of the body, and  $(x, r, \theta)$  are cylindrical coordinates. The  $\Phi_m^{(1)}$  are determined from the velocity field of the isolated wing by the boundary condition on the body. The velocity field of the wing used is one corresponding to a type stated above but with infinite aspect ratio. Therefore the analysis is not applicable to cases of small aspect ratio and long body where the Mach cones from the tips of the wing include part of the body. For the calculation of the interference on the

wing by the body, the author writes the additional perturbation potential  $\Phi'$  as

$$(A) \quad \Phi' = V_0 b \sum_m \Phi_m'(x/b, z/b) \cos \frac{1}{2} \pi m y/b,$$

where  $b$  is the semi-span,  $y$  is the span-wise direction. The  $\Phi_m'$  are determined from the velocity field of the isolated body by the boundary condition on the wing or the equivalent part of the  $(x, y)$ -plane.

For the increase of lift and moment of the body due to the presence of the wing, the author obtains the simple result that they are approximately equal to the corresponding quantities of the wing area enclosed within the body if that area were not covered by the body. The effects on the wing due to the body are likely to be larger than those on the body due to the wing. For the specific numerical example computed by the author, the ratio is roughly 2 to 1.

In the reviewer's opinion, since the interference is of the same order of magnitude as the disturbances of the isolated body and the isolated wing, the basic assumption of the author that the "interference" of interference can be neglected seems hardly justified. Furthermore, the use of  $\Phi'$  in the form of (A) is difficult to understand, as it is clear that the velocity field is not periodic in  $y$  with a period of  $4b$ .

*H. S. Tsien* (Cambridge, Mass.).

**Mirels, Harold.** Theoretical wave drag and lift of thin supersonic ring airfoils. *Tech. Notes Nat. Adv. Comm. Aeronaut.*, no. 1678, 20 pp. (1948).

Consider a thin airfoil in the shape of a ring whose ratio of chord to diameter is small. The surface lying in the Mach forecone of any point of the airfoil is nearly flat. Then, in the linearized theory, the simple proportionality between singularity strength and normal velocity, which exists for a plane wing, is approximately correct. Lift and wave drag are calculated, especially for symmetrical double-wedge profiles and cylindrical basic shapes. Comparison is made with results of Brown and Parker [Wartime Rep. Nat. Adv. Comm. Aeronaut., no. L-720 (1946)] who calculated the linearized flow outside open-nose bodies of revolution.

*W. R. Sears* (Ithaca, N. Y.).

**Sbrana, Francesco.** Sul moto di un solido immerso in un fluido. II. *Pont. Acad. Sci. Acta* 10, 297-299 (1946).

Part I appeared in *Acta Pont. Acad. Sci. Nov. Lincei* 88, 163-182 (1935).

**Moses, H. E.** The head-on collision of a shock wave and a rarefaction wave in one dimension. *J. Appl. Phys.* 19, 383-387 (1948).

The effect of a rarefaction wave upon a shock wave traveling in the opposite direction is considered in the one-dimensional case. For weak shocks in which the entropy change across the shock may be neglected, the strength of the shock may be increased by the interaction. Explicit formulas for the velocity of the shock and of the fluid on both sides are obtained in terms of the time  $t$ .

*M. H. Martin* (College Park, Md.).

**Lin, C. C., and Rubinov, S. I.** On the flow behind curved shocks. *J. Math. Physics* 27, 105-129 (1948).

The first part of this paper deals with the general characteristics of curved shock waves. A variational principle for iso-energetic flows is stated. The spatial first derivatives of the various physical quantities before and behind the shock

in plane flow are considered and it is found that they can all be related in terms of the shock inclination.

In the second part, problems of curved shocks attached to two-dimensional bodies are treated. The case of the normal shock emanating from a curved solid boundary has been treated by Emmons [Tech. Notes Nat. Adv. Comm. Aeronaut., no. 1003 (1946); these Rev. 8, 107] and Tsien [same J. 26, 69-75 (1947); these Rev. 8, 610]. Their results are confirmed for a shock having finite curvature, on a body whose directions are continuous but which may have discontinuities of curvature; namely, the ratio of curvatures before and behind the shock is fixed by the Mach number. However, by assuming that the curvature of the shock at the boundary may be infinite, the authors obtain new results for a shock attached to a body of continuous curvature: there is a critical supersonic Mach number at which a normal shock of finite curvature can form, above (below) which the shock has infinite curvature at the body, and below (above) which the shock cannot occur when the body is convex (concave). The physical meanings of this result and the Emmons-Tsien result are examined.

Finally, the flow behind detached shocks is considered. It is shown that the flow near the nose of the body and behind the nose of the shock can be calculated by power-series expansions. The scheme of calculation is developed for both plane and axi-symmetric cases. Some preliminary results are presented and additional work, as yet unpublished, by J. Dugundji, is outlined.

W. R. Sears.

**Herpin, André.** *La théorie cinétique de l'onde de choc.* Revue Sci. 86, 35-37 (1948).

The author indicates how the flow of a gas through a stationary shock wave can be treated by the methods of statistical mechanics. He illustrates by the treatment of a monoatomic gas. A large number of molecules are considered in a one-dimensional flow. Their velocities have an average forward component plus a random (thermal) component. A general intermolecular force is ascribed, and it is assumed the molecules have no internal forms of energy. The author then sets up the equations for the average transport of an arbitrary quantity  $Q$  associated with each molecule and depending only on its velocity. By specializing the form of  $Q$  he derives statistical equations equivalent to the equations for the conservation of mass, momentum and energy. Taking the flows at  $+\infty$  and  $-\infty$  to be distinct uniform flows, he derives three shock equations equivalent to the Rankine-Hugoniot equations under the obvious identification of statistical with thermodynamic parameters. By way of addendum the author considers the problem of shock wave thickness. By assuming this to be of the order of shock propagation velocity times the relaxation time of the gas he arrives at a formula for shock thickness which at high shock strengths seems to be more realistic than that derived by the classical means.

D. P. Ling.

**McVittie, G. C.** *The equations governing the motion of a perfect-gas atmosphere.* Quart. J. Mech. Appl. Math. 1, 174-195 (1948).

In most investigations of dynamic meteorology the earth's surface is regarded as a rotating disc tangential to the actual earth. For many meteorological studies this drastic simplification is unsatisfactory because of the magnitude of the atmospheric perturbations. The introduction of the customary spherical polar coordinates with the origin at the earth's center is also not considered suitable because the origin is

thus far removed from the region in which the motion is to be studied. Therefore a new system of hydrodynamic equations is developed where the origin  $O$  of the coordinate system is at the earth's surface and where the coordinates of a point  $P$  are the great circle distance from the origin  $O$ , the vertical distance from  $P$  to the ground and the angle made by the plane of the great circle through  $O$  and  $P$  with the meridian of  $O$ . In this system of equations the air is assumed as an ideal gas and as dry. The equations of motion, of continuity and of heat-transfer by bodily motion (first law of thermodynamics) are derived for this coordinate system and allowance is made for the fact that the vertical dimensions are small compared to the earth's radius. For steady horizontal motion the equations are written down in nondimensional variables.

The following types of steady horizontal motion are considered. (1) Simple rotation; it is shown that an adiabatic motion of this type is possible only if it is symmetrical and small-scale (radius not more than about 200 km). (2) Quasi-adiabatic and small-scale motion of convergence or divergence; in this case the solution does not hold near the axis of the vortex. (3) Large-scale motions; in this case no vortical motions are found, but a solution exists where the wind blows perpendicularly to the meridian through  $O$ . Since this conclusion was derived under the assumption of horizontal motion it appears possible that the difficulty can be eliminated by abandoning the assumption of strictly horizontal motion.

B. Haurwitz (New York, N. Y.).

**Davies, T. V.** *Rotatory flow on the surface of the earth.*

I. *Cyclostrophic motion.* Philos. Mag. (7) 39, 482-491 (1948).

The author considers the perturbations of a circular vortex in cyclostrophic motion. The density is assumed to be proportional to the pressure. By elimination of the velocity components and of the density a differential equation for the pressure alone is obtained which, together with the boundary condition that the vertical velocity vanishes at the ground, permits a solution of the problem. A formal solution is given for the case that undisturbed velocity increases linearly with the distance from the center. This solution shows that the vortex is subjected to a southerly force which will give it a translational motion.

B. Haurwitz (New York, N. Y.).

**Ertel, Hans.** *Die thermische und potentielle Energie atmosphärischer Aktionszentren.* Z. Meteorologie 1, 225-229 (1947).

It is shown that the thermal and potential energy of atmospheric centers of action can be computed from surface observations of the pressure without aerological data. The reason for this is that the height integrals of the atmospheric pressure which are required for the energy computation can be expressed by the pressure distribution at the ground with the aid of the hydrodynamic equations. The inertia terms which appear because of the introduction of the hydrodynamic equations are neglected when the integration over the horizontal area of the center is performed since they contain products of the velocity terms and therefore tend to zero more rapidly than the linear terms at the boundaries of the centers. The equations are applied to the Siberian anticyclone and give values in agreement with those obtained from other estimates.

B. Haurwitz.

\***Golaz, Charles.** *Étude sur la Variation de la Vitesse du Vent en Fonction de l'Altitude.* Thesis, University of Geneva, 1940. 85 pp.

After a review of conditions in the layer next the ground and of a paper by Hesselberg and Sverdrup on friction in the atmosphere the author takes into consideration the variation of the density with the elevation in an isothermal atmosphere. The problem leads to a differential equation of Frobenius' type which is solved. Next the effect of the density variation on the vertical wind distribution in an atmosphere with constant vertical temperature gradient is considered under simplifying assumptions. In both these cases the coefficient of friction is assumed to be a constant. Finally the effect of turbulence is taken into account under the assumptions (1) that the local change of the wind velocity is proportional to the second derivative of the wind velocity in the vertical (based on the Austausch equation), (2) that the coefficient of friction is an exponential function of the elevation, (3) that the Austausch coefficient is twice the coefficient of friction.

*B. Haurwitz.*

**Thompson, Philip Duncan.** The propagation of permanent-type waves in horizontal flow. *J. Meteorol.* 5, 166-168 (1948).

The author shows how, in addition to the variation  $\beta$  of the Coriolis parameter with latitude, the mean velocity of the current  $U$  and the wave length  $L$  and width  $W$  of the current, the wind shear  $dU/dy$  in the latitudinal direction affects the wave velocity  $c$ . It is found that  $c = U + (U_{\text{av}} - \beta) / \{(2\pi/L)^2 + (\pi/W)^2\}$ . The formula holds for waves of finite amplitude.

*B. Haurwitz.*

**Gerjuoy, E.** Refraction of waves from a point source into a medium of higher velocity. *Physical Rev.* (2) 73, 1442-1449 (1948).

Two media are separated by an infinite plane boundary. A point source of sound is placed in one medium and the sound field in the second is calculated. It is assumed that the wave velocity in the second medium is greater than that in the first. Absorption is neglected and the problem is solved in two ways. The author applies the ray method and then uses an exact solution of the wave equation which is made useful by an application of the method of steepest descents.

*A. E. Heins* (Pittsburgh, Pa.).

**Küssner, H. G.** Lösungen der klassischen Wellengleichung für bewegte Quellen. *Z. Angew. Math. Mech.* 24, 243-250 (1944).

The author is concerned with the calculation of the sound field of moving sources. For a uniform velocity the cases of rectilinear, circular and helical motion are treated in detail. The theory is applied to the airscrew. The method is based on the invariance of the wave equation under simple conformal transformations in  $R_4$ .

*C. J. Bouwhamp.*

**Bordoni, Piero Giorgio.** Metodi approssimati per lo studio delle sorgenti sonore. *Pont. Acad. Sci. Acta* 8, 61-66 (1944).

### Elasticity, Plasticity

\***Biezeno, C. B.** Survey of papers on elasticity published in Holland 1940-1946. *Advances in Applied Mechanics*, edited by Richard von Mises and Theodore von Kármán, pp. 105-170. Academic Press, Inc., New York, N. Y., 1948. \$6.80.

**Castoldi, L.** Deduzione variazionale delle equazioni della dinamica dei continui deformabili. *Nuovo Cimento* 5, 140-149 (1948).

**Krzywoblocki, M. Z.** On the so-called principle of least work method. II. *J. Franklin Inst.* 244, 465-469 (1947).

An expository continuation of an earlier paper [same *J. 243*, 187-204 (1947); these *Rev.* 9, 164] dealing with the relation of the earlier ideas to the theorem of minimum potential energy and with techniques applying the theorem of least work. *F. B. Hildebrand* (Cambridge, Mass.).

**Timpe, A.** Torsionsfreie achsensymmetrische Deformation von Umdrehungskörpern und ihre Inversion. *Z. Angew. Math. Mech.* 28, 161-166 (1948).

The author derives various relations in the theory of rotationally symmetric deformations of elastic bodies which follow from a consideration of the differential equations written on the basis of spherical coordinates. Various types of solutions for a stress function which satisfies the biharmonic equation are indicated. The main result is the observation that, similarly to what is known for plane stress, new solutions for the rotationally symmetric problem may be obtained by the process of inversion. *E. Reissner*.

**Schmeidler, W.** Über die Wärmespannungen in einem Körper. *Z. Angew. Math. Mech.* 28, 54-59 (1948).

The problem of thermal stresses in an elastic body with vanishing surface stresses is formulated as a system of four simultaneous linear integral equations for the divergence and for the components of the curl of the displacement vector.

*E. Reissner* (Cambridge, Mass.).

**Schmeidler, W.** Zurückführung der Wärmespannungen in einem elastischen Körper auf ein Knick-Biegungsproblem. *Z. Angew. Math. Mech.* 28, 92-93 (1948).

A decomposition of the displacement vector is given which reduces the problem of finding the displacements in an elastic body subject to a given steady temperature distribution with zero surface stresses to that of finding the displacements for the case of zero temperature but with a given distribution of surface stresses. *H. J. Greenberg*.

**Cattaneo, Carlo.** Azioni elastico-dissipative a ciclo d'istresi ellittico. *Pont. Acad. Sci. Acta* 9, 139-156 (1945).

**Volterra, Vito.** Energia nei fenomeni elastici ereditari. *Pont. Acad. Sci. Acta* 4, 115-128 (1940).

The author proves that in an elastic body with after-effect only part of the work done by the external forces is transformed into kinetic or elastic energy; the remainder is dissipated.

*W. Prager* (Providence, R. I.).

**Gerasimov, A. N.** A generalization of linear laws of deformation and its application to problems of internal friction. *Akad. Nauk SSSR. Prikl. Mat. Meh.* 12, 251-260 (1948). (Russian)

The author applies the technique of operational calculus to the solution of one-dimensional problems concerning a Boltzmann material (flow between parallel plates and flow between coaxial cylinders).

*W. Prager*.

**Philippidis, A.** Eine Beziehung zwischen der nichtlinearen Elastizitätstheorie und der Verfestigungstheorie von Roš-Eichinger-Schmidt. *Z. Angew. Math. Mech.* 25/27, 31-32 (1947).

The author points out that the stress-strain relations which Roš, Eichinger and Schmidt have proposed for strain-

hardening materials can also be interpreted as nonlinear stress-strain relations for an elastic material. [The author's definition of the term "elastic" is somewhat formal: it does not imply the existence of an elastic potential. However, the result of the present paper remains valid even if a more conventional definition of the term "elastic" is used.]

W. Prager (Providence, R. I.).

Reiner, M. Elasticity beyond the elastic limit. *Amer. J. Math.* 70, 433-446 (1948).

The paper is concerned with the relation between stress and recoverable elastic strain in a material which has been stressed beyond the elastic limit. [The author uses the terms "deformation" and "strain" for the total strain and its recoverable part, respectively; this terminology conflicts with the customary use of these terms and has, therefore, not been adopted in this review.] Various measures of finite elastic strain are discussed and the general structure of the relation between stress and elastic strain in an isotropic material is established independently of the measure of strain.

W. Prager (Providence, R. I.).

Prager, W., and Synge, J. L. Approximations in elasticity based on the concept of function space. *Quart. Appl. Math.* 5, 241-269 (1947).

The results obtained in the present paper concern the problem of the linear theory of elasticity (without body forces) and are based on the assumption of a positive definite strain energy function quadratic in the components of stress. The strain energy function provides a metric in function space in which a point (or vector) represents a state of stress. The aim is to obtain approximate solutions of boundary value problems, with calculable errors which are measured in terms of distance in function space. The authors first discuss the notion of vectors in function space. The natural state of stress in a body which must satisfy equilibrium equations, compatibility equations and boundary conditions defines a vector  $S$  with square of length equal to twice the strain energy of the body. Besides  $S$  there are introduced vectors  $S^*$  and  $S''$ , corresponding to states of stress which satisfy some but not all the conditions imposed on  $S$ , in such a way that  $S$  is the only state of stress which can at the same time be both  $S^*$  and  $S''$ . The definitions of  $S^*$  and  $S''$  depend on the form of the boundary conditions which are prescribed. The final result consists in every case in certain upper and lower bounds for an unknown  $S$  in terms of assumed states  $S^*$  and  $S''$ .

E. Reissner.

Synge, J. L. The method of the hypercircle in function-space for boundary-value problems. *Proc. Roy. Soc. London. Ser. A.* 191, 447-467 (1947).

The author applies the notions of function space as defined in an earlier paper by Prager and Synge [cf. the preceding review] on boundary value problems in elasticity to the Dirichlet and Neumann problems in Riemannian  $N$ -space, to vibration problems, and to slightly more general elastostatic problems. As before, the boundary value problem is split into two relaxed problems such that a common solution of the relaxed problems is the solution of the original problem. Next it is decided what set of functions shall correspond to a vector in function space. Finally a suitable definition of the scalar product of two vectors in function space gives a metric in this space. From this point on the procedure is to locate the solution of the problem on or inside a hypercircle in function space, where center

and radius of the hypercircle are determined by means of suitable solutions of the relaxed problems.

E. Reissner (Cambridge, Mass.).

Synge, J. L. The method of the hypercircle in elasticity when body forces are present. *Quart. Appl. Math.* 6, 15-19 (1948).

The results of Prager and the author [cf. the second preceding review] are extended to include problems with body forces present.

E. Reissner (Cambridge, Mass.).

Greenberg, H. J., and Truell, Rohn. On a problem in plane strain. *Quart. Appl. Math.* 6, 53-62 (1948).

In this paper the authors consider the following problem of plane strain for a rectangular region. Two opposite edges are assumed to be free of stress while the other two edges are loaded in such a way that there are no tangential displacements and uniform normal displacements. (The corresponding problem for rotational symmetry is of obvious practical significance). The object of the work is to obtain upper and lower bounds for load-deflection ratio for the specimen, as a function of length-width ratio. This is done, using the recently developed method of Prager and Synge [cf. the third preceding review] for the determination of bounds on the strain energy of deformation. The steps in applying this method are indicated in some detail.

E. Reissner (Cambridge, Mass.).

Prager, W. Theory of plastic flow versus theory of plastic deformation. *J. Appl. Phys.* 19, 540-543 (1948).

This paper presents a summary of the essential features of typical theories of plastic flow and plastic deformation and defines loading and unloading in terms of the octahedral shearing stress. Compressibility and viscosity are neglected. It is then shown that in theories of plastic flow the stress-strain relations for both loading and unloading predict the same strain change in the limiting case of neutral change of stress, while in theories of plastic deformation this requirement is not fulfilled.

F. B. Hildebrand.

Prager, William. The stress-strain laws of the mathematical theory of plasticity—a survey of recent progress.

*J. Appl. Mech.* 15, 226-233 (1948).

Typical stress-strain laws of flow and deformation types are discussed with particular reference to the conditions of continuity and uniqueness which these laws must fulfill if they are to make sense physically. Alternative forms of some of these laws are presented and conditions are discussed under which different laws yield identical results. It is shown how the laws may be generalized so as to fit test data more readily. Methods of integration are indicated and the use of variational principles is stressed.

F. B. Hildebrand (Cambridge, Mass.).

Prager, W. Problem types in the theory of perfectly plastic materials. *J. Aeronaut. Sci.* 15, 337-341 (1948).

A structural example is used to motivate a classification of typical problems in the theory of perfectly plastic materials. The applicability of appropriate existing minimum principles in certain cases when the history of the loading is either known or irrelevant is pointed out. With reference to other cases, a discussion is given of conditions under which the structure considered will approach a state of residual stress together with elastic stresses corresponding to further loading, and a minimum principle relevant to such cases is conjectured. Finally, the problem of structural

stability in the plastic range is discussed and is shown to differ in formulation from the corresponding problem in the elastic range. *F. B. Hildebrand* (Cambridge, Mass.).

\***Prager, W.** *Discontinuous solutions in the theory of plasticity.* Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, pp. 289-300. Interscience Publishers, Inc., New York, 1948. \$5.50.

According to Saint Venant's theory of plasticity the stress distribution is statically determinate when the plastic medium is in a state of plane flow under the action of given boundary stresses. The governing partial differential equations allow discontinuities of the first derivatives of the stress components across the lines of maximum shearing stress which coincide with the characteristics of the differential equations, in analogy with "sonic discontinuities" in the dynamics of compressible fluids. The present paper first reviews the theory underlying these facts and then proceeds to investigate stronger discontinuities analogous to such phenomena as "shock fronts." It is shown that the normal stress transmitted across a surface element which is parallel to the generators of a cylindrical discontinuity surface and normal to that surface may be discontinuous. In consequence it follows that the shear lines may possess discontinuities in curvature. Explicit expressions for the magnitudes of these jumps are obtained and are applied to the treatment of a specific example.

*F. B. Hildebrand.*

**Prager, W.** *On the interpretation of combined torsion and tension tests of thin-wall tubes.* Tech. Notes Nat. Adv. Comm. Aeronaut., no. 1501, 11 pp. (1948).

Combined torsion and tension tests of thin-walled tubes are of frequent use in checking the various theories of plasticity. The author points out the fact that when the ratio of axial load to impressed torque is held constant throughout a test various such theories furnish identical predictions. More general tests of this sort are proposed and suggestions are given for the interpretation of such tests.

*F. B. Hildebrand* (Cambridge, Mass.).

**Cicala, P.** *Sull'analisi delle piccole deformazioni nel campo elastoplastico.* Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 3, 325-329 (1947).

The author reviews the stress-strain laws of Mises, Hencky and Reuss and indicates two schemes of verifying these laws experimentally. In the first type of experiment, two identical thin-walled tubes with parallel axes and coplanar end surfaces are built in at, say, the left-hand ends and subjected to equal and opposite torques at the right-hand ends. When these torques produce plastic yielding in the tubes, the right-hand ends are clamped in a single rigid head. The beam consisting of the two tubes which are under initial stress is then subjected to flexure in a plane normal to that containing the axes of the tubes. The three stress-strain laws predict different values of the initial slope of the curve showing bending moment versus curvature. [The realization of this scheme will meet with considerable practical difficulties. A similar but less complicated experiment involving combined tension and torsion of a thin-walled tube was suggested by the reviewer in the paper reviewed above.] In the second experiment suggested by the author, a thin-walled tube closed at the ends is first subjected to longitudinal tension. When plastic yielding occurs, the longitudinal displacements of the end surfaces are kept constant and the tube is subjected to internal pressure.

The three stress-strain laws predict different values for the initial slope of the curve showing hoop stress versus circumferential extension. *W. Prager* (Providence, R. I.).

**Handelman, G. H., and Prager, W.** *Plastic buckling of a rectangular plate under edge thrusts.* Tech. Notes Nat. Adv. Comm. Aeronaut., no. 1530, 97 pp. (1948).

The authors begin by discussing the advantages of a theory of plastic flow over one of plastic deformation, particularly with reference to desirable continuity of stress-strain relations in transition between loading and unloading. An explicit theory of the preferred type is developed and compared with other existing theories. Next appropriate differential equations and boundary conditions governing the plastic buckling of a simply compressed plate are formulated. In particular, it is shown that here the plate must be considered as anisotropic, the anisotropy being a function of the compressive stress. As a first specific application, the plastic buckling of a narrow strip is investigated and the results are found to tend in the limit toward those contained in the Engesser-Kármán theory of buckling of beams beyond the elastic limit, so long as the material is assumed to be incompressible. In the case of buckling of a simply supported rectangular plate, explicit results are again obtained and compared with the results of other theories. Torsional buckling of cruciform sections is considered as a third example. An energy method generalizing that relevant to elastic buckling is then presented, and its usefulness in obtaining approximate solutions is illustrated in the case of buckling of a cruciform section. A number of numerical results are presented in curve form.

*F. B. Hildebrand* (Cambridge, Mass.).

**Stowell, Elbridge Z.** *A unified theory of plastic buckling of columns and plates.* Tech. Notes Nat. Adv. Comm. Aeronaut., no. 1556, 31 pp. (1948).

The problem of column stability is concerned with the determination of the smallest axial load for which the column can assume a bent as well as a straight equilibrium configuration. The classical treatment of column stability in the plastic range, however, is concerned with a more restricted problem, namely the determination of the smallest axial load such that the column can pass from the straight to a bent equilibrium configuration while under the action of this load. In the elastic range, the difference between these two problems disappears, because the exterior forces necessary to maintain the system in a certain configuration depend only on this configuration and not on the preceding configurations of the system. In the plastic range, however, the history of deformation is of importance, and the two problems formulated above have different solutions. As far as the plastic buckling of rectangular plates under edge thrusts is concerned, the second problem was treated by A. A. Ilyushin [Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 8, 337-360 (1944); these Rev. 7, 43]. The present paper treats the practically more important first problem, using Ilyushin's stress-strain law. [Unfortunately, this stress-strain law of the deformation type is open to strong objections (see, for instance, the reviewer's remarks in the review mentioned above). G. Handelman and W. Prager treated the second problem using a stress-strain law of the flow type; cf. the preceding review. It would seem desirable to apply the same stress-strain law to the problem treated in the present paper.]

*W. Prager* (Providence, R. I.).

Hodge, P., and Prager, W. A variational principle for plastic materials with strain-hardening. *J. Math. Physics* 27, 1-10 (1948).

After introducing a suitable definition of loading and unloading of a plastic material with strain-hardening, the authors establish the following variational principle. The volume integral of  $\dot{\sigma}_{ij}\dot{\epsilon}_{ij}$  over the entire body is less for the actual stress rates than it is for any system of artificial stress rates which satisfy equilibrium and boundary conditions, and which either are indefinitely near the actual stress rates, or constitute unloading in all regions of the body where the actual stress rates constitute unloading. (Here  $\dot{\sigma}_{ij}$  and  $\dot{\epsilon}_{ij}$  represent the rates of stress and strain.) An extension to certain mixed boundary value problems is also indicated. The results apply only to cases when loading from the stress-free natural state is followed by at most one unloading.

F. B. Hildebrand (Cambridge, Mass.).

Winzer, Alice, and Prager, W. On the use of power laws in stress analysis beyond the elastic range. *J. Appl. Mech.* 14, A-281-A-284 (1947).

In a recent paper [Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 10, 347-356 (1946); these Rev. 8, 240] A. A. Ilyushin conjectured that, regardless of the particular stress-strain law used, a gradual increase in the loading of a plastic material should correspond to a proportionate increase in stresses. The authors show by means of an explicit example that this may not be the case unless the stress-strain law is such that the secant shear modulus is strictly proportional to a power of the octahedral shearing stress.

F. B. Hildebrand (Cambridge, Mass.).

Carrier, G. F. The extrusion of plastic sheet through frictionless rollers. *Quart. Appl. Math.* 6, 186-192 (1948).

This paper deals with the application of the Saint Venant-Mises theory of slow plane plastic flow to the analysis of deformations associated with the extrusion of a plastic sheet through equal fixed frictionless cylindrical rollers with parallel axes. In the case when the thickness of the sheet is small with respect to the radii of the cylinders it is shown that the results of a simple one-dimensional theory are in satisfactory agreement with approximate results obtained from the precise theory. For this purpose, use is made of the method of characteristics to replace the precise equations by a pair of nonlinear integrodifferential equations, the approximate solution of which is stated to lead to results which justify the use of the simpler theory in the cases described.

F. B. Hildebrand (Cambridge, Mass.).

Siebel, E. The application to shaping processes of Hencky's laws of equilibrium. *J. Iron and Steel Inst.* 155, 526-534 (1947).

Siebel, E. Anwendung der Henckyschen Sätze über das Gleichgewicht in plastischen Körpern auf die technischen Formgebungsverfahren. *Ing.-Arch.* 16, 164-172 (1948).

Using the theorems of Hencky and Prandtl concerning the geometric properties of the net of slip lines in problems of plane plastic strain, the author presents a qualitative discussion of important forming processes (forging, rolling, drawing and extruding). [While the slip line patterns used by the author satisfy the static boundary conditions, they violate, in most cases, equally important kinematic boundary conditions.]

W. Prager (Providence, R. I.).

Hill, R. A theory of the yielding and plastic flow of anisotropic metals. *Proc. Roy. Soc. London. Ser. A.* 193, 281-297 (1948).

The author uses the plastic potential introduced by R. von Mises [Z. Angew. Math. Mech. 8, 161-185 (1928)] to construct stress-strain laws for polycrystalline metals which have become orthotropic through previous plastic strain. These stress-strain laws are applied to the analysis of the following problems: (1) the necking of a strip under tension, (2) the effect of the anisotropy developed in a torsion test and (3) the earing of deep-drawn cups. The results obtained under (1) are found to agree with experiments of Körber and Hoff.

W. Prager.

Sokolovskii, V. V. On a form of representation of the components of stress in the theory of plasticity. *Doklady Akad. Nauk SSSR (N.S.)* 61, 223-225 (1948). (Russian)

The author gives a parametric representation of the stress components with respect to an arbitrary set of rectangular axes in terms of the Eulerian angles of the principal axes of stress and three further parameters. One of these is proportional to the octahedral shearing stress; the other two do not seem to have immediate physical significance. The special cases of plane strain, plane stress, torsion, axially symmetric plane strain and axially symmetric plane stress are discussed.

W. Prager (Providence, R. I.).

Ševčenko, K. N. The elastic-plastic state due to a concentrated force applied to a half-plane. *Doklady Akad. Nauk SSSR (N.S.)* 61, 29-30 (1948). (Russian)

The paper is concerned with the stresses and strains set up in the elastic-plastic half-plane  $x > 0$  by a concentrated force which is applied at the origin and acts along the positive  $x$ -axis. The author assumes that the well-known elastic stress distribution remains valid even when part of the material has become plastic. Using Hencky's stress-strain relations for a strain-hardening material, he then proceeds to show that the plastic strains obtained on the basis of this assumption satisfy the equation of compatibility.

W. Prager (Providence, R. I.).

Ševčenko, K. N. A concentrated force applied to a half plane (elastic-plastic problem). *Akad. Nauk SSSR. Prikl. Mat. Meh.* 12, 385-388 (1948). (Russian)

Cf. the preceding review.

Taylor, G. I. The formation and enlargement of a circular hole in a thin plastic sheet. *Quart. J. Mech. Appl. Math.* 1, 103-124 (1948).

The paper is concerned with the following idealized version of the piercing of a flat sheet by a conical-headed bullet: the sheet is supposed to have a cylindrical hole which, starting from zero radius, is enlarged by gradually increasing radial pressure; the problem is treated as one in plane stress, and the deformation is assumed to be symmetric with respect to the middle plane of the sheet. Under these conditions the configuration when the hole has the radius  $b$  will be similar to that when the hole has the radius  $b_1$  except that the radii where any given thickness occurs will be changed in the ratio  $b_1/b$ . With the view to further simplification, the sheet material is assumed to be isotropic and Mohr's yield criterion is used. The material is found to be elastic for  $r > 3.64b$ . In the annulus  $2.21b < r < 3.64b$ , the material is plastic, the hoop stress being tension, and the plastic strains being comparable in magnitude to those occurring in the elastic zone. For  $r < 2.21b$ , too, the material

is plastic, but the hoop stress is compression and the plastic strains are large. At the edge of the hole the sheet thickens to 2.61 times its original thickness.

Quite aside from its relation to the ballistic problem mentioned above, the present investigation is of interest because it represents one of the very few cases in the mathematical theory of plasticity where the complete strain-history of each element of the material has been adequately taken into account. This is necessary here because the ratios of the principal stresses of each element vary as the deformation develops. The analysis of a problem of this kind cannot be based on a relationship between stress and total deformation [theory of plastic deformation], but only on a relationship between the infinitesimal increments of stress and strain [theory of plastic flow]. The author points out that "eminent authorities have not always appreciated this point and have consequently published erroneous solutions of problems in which the stresses and total plastic strains have been related as though the stress distribution had been constant during the deformation when in fact it had not."

W. Prager (Providence, R. I.).

Colonnetti, Gustavo. *Al di là dei limiti della teoria classica dell'elasticità*. Pont. Acad. Sci. Acta 5, 159-166 (1941).

The author takes issue with recent Italian papers on plasticity [Locatelli, Finzi, Pastori]. He states his belief that it is not possible to obtain a satisfactory theory of plasticity by merely replacing Hooke's law by a nonlinear relation between stresses and strains and points out that, as far as the relation between the infinitesimal changes of stress and strain are concerned, it is not the assumption of linearity which must be abandoned but the assumption of the homogeneity of the material.

W. Prager.

Colonnetti, Gustavo. *Deformazioni plastiche e deformazioni viscose*. Pont. Acad. Sci. Acta 6, 217-224 (1942).

The paper contains a brief qualitative discussion of creep and its influence on the stresses in structures.

W. Prager (Providence, R. I.).

Feinberg, S. M. *The principle of limiting stress*. Akad. Nauk SSSR. Prikl. Mat. Meh. 12, 63-68 (1948). (Russian)

The paper is concerned with applications of a heuristic principle which, in the engineering literature, is sometimes referred to as the principle of the intelligence of the material. Defining an "admissible state of stress" in an elastic-plastic structure as a state of stress which satisfies all conditions of equilibrium and nowhere violates the yield condition, the author formulates his principle as follows: if in a structure under given loads there exist a number of admissible states of stress, then one of these is actually realized in the structure. (This principle is supposed to hold independently of the manner in which the loads have been brought up to the given values.) With each admissible state  $S$  a positive multiplier  $\lambda$  is associated; this is defined as the greatest positive number by which the stresses of  $S$  can be multiplied without violating the yield condition. If the validity of the principle of limiting stress is accepted, the safety factor of the structure under the given loads is obtained as the maximum of  $\lambda$  for all admissible states of stress. If, for example, the octahedral shearing stress  $\sigma_0$  is accepted as the yield criterion, the "deviation of a given state of stress from zero" is defined as the maximum value of  $\sigma_0$  for this state of stress. The state of stress which furnishes the safety factor is then characterized as that admissible state of stress

which possesses the smallest deviation from zero. The author indicates briefly how upper and lower bounds could be obtained for the safety factor, but does not give any examples.

W. Prager (Providence, R. I.).

Richard, U. *Dilatazione di una fune pesante sospesa a due estremi fissi*. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 3, 321-325 (1947).

L'auteur calcule les composantes de la réaction dans un des points de suspension d'une corde, étant données les coordonnées des deux points fixes et la longueur de la corde; et la variation de ces composantes pour une dilatation de la corde.

B. Levi (Rosario).

Galin, L. A. *An estimate for the displacement in spatial contact problems of the theory of elasticity*. Akad. Nauk SSSR. Prikl. Mat. Meh. 12, 241-250 (1948). (Russian)

The paper is concerned with the problem of contact of a rigid punch with an elastic half-space. In section 1, the base of the punch is bounded by the surface  $z=f(x, y)$  and the section of the punch, by the plane  $z=0$ , is an ellipse  $D$ . Thus  $f(x, y)$  is the displacement of the punch in the  $z$ -direction. The punch is acted on by the force  $P$  directed along the  $z$ -axis and by moments  $M_x$  and  $M_y$  with respect to the  $x$ - and  $y$ -axes. The state of stress is determined by a harmonic function  $\varphi(x, y, z)$  such that  $\varphi=f(x, y)$  in  $D$ ,  $\partial\varphi/\partial z=0$  outside  $D$ ,  $\varphi(x, y, z)\rightarrow 0$  as  $x^2+y^2+z^2\rightarrow\infty$ . The author determines  $\varphi$  in a series of Lamé's functions and calculates the displacement in the  $z$ -direction and the rotation of the punch relative to the  $x$ - and  $y$ -axes in terms of  $P$ ,  $M_x$  and  $M_y$ .

In section 2 the base of the punch is assumed to be plane and the cross-section enclosing the area  $A$  of contact is bounded by an arbitrary curve  $L$ . Let  $a$  be the major semi-axis and  $e$  the eccentricity of an ellipse circumscribing  $L$ ; then for a given displacement  $\delta$  of the punch, the impressed force  $P$  is bounded by

$$2\delta E(1-\sigma^2)^{-1}(A/\pi)^{1/2} < P < \delta E(1-\sigma^2)^{-1}\pi a/F(\pi/2, e),$$

where  $E$  is Young's modulus,  $\sigma$  is Poisson's ratio, and  $F(\pi/2, e)$  is the complete elliptic integral of the first kind.

I. S. Sokolnikoff (Los Angeles, Calif.).

Galin, L. A. *On the pressure of a solid body on a plate*. Akad. Nauk SSSR. Prikl. Mat. Meh. 12, 345-348 (1948). (Russian)

A rigid body whose surface is determined by the equation  $z=Ax^2+By^2$  is brought in contact with a thin clamped circular elastic plate of radius  $R$ . The point of initial contact is the center of the plate  $x=0, y=0$ . If the force  $P$ , acting along the  $z$ -axis, is applied to the body, what is the shape of the area of contact? It is shown that the area of contact is an ellipse provided that its linear dimensions are small compared with  $R$  and with the radii of curvature of the surface. The situation in this case is similar to that occurring in the problem of contact of a rigid ellipsoid with an elastic half-space considered by H. Hertz.

I. S. Sokolnikoff (Los Angeles, Calif.).

Ghosh, S. *On a new function-theoretic method of solving the torsion problem for some boundaries*. Bull. Calcutta Math. Soc. 39, 107-112 (1947).

The author solves the torsion problem for a cross section whose boundary consists partly of one straight line. The region is conformally mapped into a semicircular area in the positive half plane with the straight boundary becoming the bounding diameter. By Schwarz's principle of reflection

the torsion function is continued analytically to the other half of the unit circle so that the problem becomes the determination of a function analytic within the entire unit circle such that it assumes prescribed values on the circumference. Application is made to the semicircular section and to a semiloop of the lemniscate, arriving at previously known results.

D. L. Holl (Ames, Iowa).

**Belluzzi, Odone.** *Sul calcolo delle travi ad arco circolare caricate normalmente al piano dell'asse.* Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) 2, 55-63 (1946).

**Belluzzi, Odone.** *Sulla stabilità dei tubi a parete sottile compresi uniformemente secondo l'asse.* Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) 3, 33-36 (1947).

**Teofiliato, Pietro.** *Trave a sbalzo con incastro non rigido e caratteristiche d'inerzia esponenziali.* Pont. Acad. Sci. Comment. 8, 585-607 (1944).

**Liebold, Rudolf.** *Die Durchbiegung einer beidseitig fest eingespannten Blattfeder.* Z. Angew. Math. Mech. 28, 247-249 (1948).

**Rieve, J.** *Die Spannungverteilung zwischen Gurt- und Stegblech unter der örtlichen Lasteinteilung beim I-Querschnitt.* Z. Angew. Math. Mech. 28, 210-217 (1948). (German. Russian summary)

Es werden die Spannungen bestimmt, die bei I-förmigen Querschnitten dadurch entstehen, dass eine Einzellast am oberen Flansch einzuleiten ist. Die Aufgabe wird zurückgeführt auf die Untersuchung der Halbebene mit einem aufgesetzten Gurt als Begrenzung und mit einer Einzelkraft als Belastung. Als Ergebnis wird eine Gebrauchsformel für die Spannungsverteilung zwischen Gurt und Halbebene angegeben.

*Author's summary.*

**Lin, T. H., and Ching, K. S.** *Buckling of a column with elastic supports.* J. Aeronaut. Sci. 15, 347-350 (1948).

The authors consider the buckling of a column with  $n$  equally spaced elastic supports supplying linear but not rotational restraints. The load producing buckling is compressive. It is assumed that the deflection of the column can be represented with sufficient accuracy by a finite Fourier sine series with  $2n-1$  terms. The method of minimum energy is then used to obtain the critical load and the corresponding deflection. The results obtained agree closely with those presented by S. P. Timoshenko [Theory of Elastic Stability, McGraw-Hill, New York, 1936, pp. 100-112].

G. E. Hay (Ann Arbor, Mich.).

**Gran Olsson, R.** *Knickung einer axial gedrückten, um ihren Mittelpunkt rotierenden Speiche.* Z. Angew. Math. Mech. 24, 224-233 (1944).

Consider a rigid ring containing an elastic spoke along a diameter with the spoke in compression when the system is at rest. Now let the ring and spoke rotate about the ring center at a fixed rate. The question investigated in this paper concerns the buckling of the spoke. It is found that the differential equation admits Whittaker functions (or, in particular cases, Bessel and Hermite functions) as its eigenfunctions. The author finds it convenient, however, to find the eigenvalues by the conventional variational (Rayleigh) procedure. [The Fredholm equation technique would have given the same accuracy with less labor.] The numerical results are tabulated.

G. F. Carrier (Providence, R. I.).

**Gran Olsson, R.** *Über die Knickung der Kreisringplatte von veränderlicher Dicke.* Ing.-Arch. 12, 123-132 (1941).

The author investigates the differential equation of the problem of buckling of a circular ring plate of variable thickness when the inner edge is free and the outer edge is acted upon by radial stresses. For thickness variations of the form  $cr^n$  he determines the values of  $n$  for which the solutions become confluent hypergeometric functions. The corresponding stability conditions are obtained for various conditions of support. Numerical evaluation of the results is left for a subsequent publication.

E. Reissner.

**Reutter, F.** *Über die Stabilität dreischichtiger Stäbe und Platten, deren mittlere aus einem Leichtstoff bestehende Schicht einen in Dickenrichtung veränderlichen Elastizitätsmodul hat. II. (Die optimalen Bemessungsgrößen).* Z. Angew. Math. Mech. 28, 132-142 (1948).

On the basis of the work reported in Part I [same vol., 1-12 (1948); these Rev. 9, 396] the author investigates the problem of the sandwich plate with highest buckling load for given weight of the plate, using assumptions for which reference must be made to the paper itself. The extremum problem is fully discussed numerically and the conclusion is obtained that a core with appropriately varying elastic modulus may result in an overall weight saving compared with what happens for uniform core material.

E. Reissner (Cambridge, Mass.).

**Pflüger, A.** *Zum Beulproblem der anisotropen Rechteckplatte.* Ing.-Arch. 16, 111-120 (1947).

The present paper contains a stability theory for thin elastic plates which are reinforced by closely spaced stiffeners in such a way that the middle surface of the plate need not be a plane of symmetry. The composite structure may then be considered as a nonhomogeneous orthotropic plate constructed by joining face to face two elastic orthotropic layers of different thickness and with different elastic properties. Retaining the assumption that normals to the undeformed middle surface are deformed into normals to the deformed middle surface, the author obtains a system of simultaneous differential equations for the components of displacement  $u, v, w$  of the middle surface which take the place of the well-known single differential equation for  $w$  for plates in which middle surface and plane of symmetry coincide. The equations of the problem are solved for a simply supported rectangular plate subjected to uniform edge thrust and the resultant solution is discussed qualitatively and quantitatively. The author also lists the variational problem corresponding to his differential equations, thereby providing a tool for obtaining approximate solutions in those cases where the boundary conditions of the problem make exact solutions prohibitively difficult.

E. Reissner.

**Kirste, L.** *Eine Erweiterung der Steifigkeitsmethode.* Österreich. Ing.-Arch. 2, 226-229 (1948).

Discussion of the analogy between the buckling of an elastically supported beam and the buckling of rectangular plates. Evaluation of the stability determinant for the case of two infinitely long plates joined in such a way that a V-section is formed, and with the two infinite edges of the plate combination simply supported.

E. Reissner.

**Federhofer, Karl.** *Die dünne Kreisringplatte mit grosser Ausbiegung.* Z. Angew. Math. Mech. 24, 189-194 (1944).

This paper is concerned with the following problem in the theory of finite deflections of thin elastic plates. A cir-

cular ring plate is simply supported along the outer edge and subjected to an axi-symmetrical distribution of transverse edge loads along the inner edge. A solution in closed form of this problem has previously been obtained by E. Schwerin [Z. Techn. Phys. 10, 651-659 (1929)], under the assumption of negligible bending stiffness of the plate. In the present paper approximate solutions are obtained without omission of the bending terms in the differential equations of the problem. Numerical calculations for a series of cases show the appreciable influence of bending stiffness on deflection and stresses in the plate.

E. Reissner (Cambridge, Mass.).

**Goriup, K.** Die dreiseitig gelagerte Rechteckplatte. I. Ing.-Arch. 16, 77-98 (1947).

This paper contains comprehensive discussions of the solutions for transverse bending of rectangular plates with three edges simply supported and one edge free, for various practically interesting conditions of loading.

E. Reissner (Cambridge, Mass.).

**Goriup, K.** Die dreiseitig gelagerte Rechteckplatte. II. Ing.-Arch. 16, 153-163 (1948).

In continuation of his earlier work reviewed above the author now considers problems for which two or three of the previously simply supported edges are clamped. An explicit solution is now no longer possible and the author makes use of the procedure previously used by H. Hencky in his well-known dissertation for the rectangular plate with all four edges clamped. E. Reissner (Cambridge, Mass.).

**Pucher, A.** Über die Singularitätenmethode an elastischen Platten. Ing.-Arch. 12, 76-100 (1941).

The author obtains influence functions for various problems of transverse bending of thin plates, that is, singular solutions of various kinds of the two-dimensional biharmonic equation. Numerical as well as analytical results are included.

E. Reissner (Cambridge, Mass.).

**Riz, P. M.** On the asymptotic integration of the equations of the theory of elasticity with applications to a plate and a disk of variable thickness. Akad. Nauk SSSR. Prikl. Mat. Meh. 12, 349-352 (1948). (Russian)

This note contains a sketch of the method of approximate integration of the three-dimensional equilibrium problem of elasticity for "thin" bodies. The bodies under consideration are symmetric with respect to the  $(xy)$ -plane and are determined by an equation of the form  $y = \beta f(x, z)$ , where  $\beta$  is a small thickness parameter. The displacements  $u, v$ , and  $w$  are assumed to have the expansions of the type  $u = u_0 + \sum_{n=1}^{\infty} u_n \beta^n$ . If one introduces the variable  $\eta = \beta y$ , the equations of Lamé in terms of  $x, y$ , and  $z$  assume the form typified by

$$\mu \left( \nabla^2 u + \frac{1}{\beta^2} \frac{\partial^2 u}{\partial \eta^2} \right) + (\lambda + \mu) \left( \frac{\partial \Lambda}{\partial x} + \frac{1}{\beta} \frac{\partial^2 v}{\partial x \partial \eta} \right) = X,$$

where  $\nabla^2 = \partial^2 / \partial x^2 + \partial^2 / \partial z^2$  and  $\Lambda = \partial u / \partial x + \partial w / \partial z$ . The substitution of expansions for  $u, v$ , and  $w$  in such equations and the usual argument of the method of small parameters yields an infinite system of differential equations for the determination of  $u, v, w$ . The boundary conditions on the displacements are determined in a similar way. As an illustration of the method, the problem of deformation of a disk whose surface is given by  $y = \beta f((x^2 + z^2)^{1/2})$  and subjected to radial body forces is discussed. The results coincide with known equations for displacements in rotating turbine disks.

I. S. Sokolnikoff (Los Angeles, Calif.).

**Libove, Charles, and Batdorf, S. B.** A general small-deflection theory for flat sandwich plates. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 1526, 53 pp. (1948).

The authors consider a sandwich plate of uniform thickness as equivalent to a simple orthotropic plate characterized by seven physical constants, four of which are related by a reciprocal law. Tests for evaluating these constants experimentally are presented in an appendix. The effects of transverse shear on small deflections of such a plate are partially taken into account by retaining the conventional assumption that cross sections initially plane and normal to the middle surface remain plane after deformation, but permitting such sections to rotate in such a way that they need not remain normal to the deformed middle surface. The basic differential equations and boundary conditions relevant to these assumptions are then obtained both by geometrical considerations and by use of variational methods. In the special case of isotropy parallel to the middle surface, the results reduce to the conventional ones if transverse shear effects are neglected by assuming infinite shear stiffness in the transverse direction. However, if finite shear stiffness is assumed the results differ from those given by E. Reissner [Quart. Appl. Math. 5, 55-68 (1947); these Rev. 8, 547] in this special case, in consequence of the neglect of transverse normal stress effects in the paper under review. As was pointed out by Reissner, the inclusion of transverse shear effects is found to lead to a clarification of classical difficulties regarding the number and nature of the relevant boundary conditions. F. B. Hildebrand.

**Sen, Bibhutibhusan.** Direct determination of stresses in thin elastic plates having cavities of different shapes. Bull. Calcutta Math. Soc. 39, 113-118 (1947).

**Giovannoni, Renato.** Il calcolo della piastra anulare spessa. Pont. Acad. Sci. Acta 10, 245-266 (1946).

**Zanaboni, Osvaldo.** Il principio di reciprocità delle tensioni tangenziali nelle lastre a doppia curvatura, e le sue immediate conseguenze. Ann. Mat. Pura Appl. (4) 25, 287-311 (1946).

**Belluzzi, Odone.** Il calcolo semplificato delle lastre a doppia curvatura. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) 3, 37-46 (1947).

**Eggwertz, Sigge.** Theory of elasticity for thin circular cylindrical shells. Summary of development and use in European structural engineering. Acta Polytech., no. 13 = Trans. Roy. Inst. Tech. Stockholm 1947, no. 9, 26 pp. (1947).

A summary is given of the development in Europe of modern circular cylindrical shells, especially the use of such shells in structural engineering and their design according to the mathematical theory of elasticity. Various procedures are described for solving the differential equations employed in the analytical method of calculation.

From the author's summary.

**Vekua, I. N.** On the theory of shallow elastic shells. Akad. Nauk SSSR. Prikl. Mat. Meh. 12, 69-74 (1948). (Russian)

It was shown by V. Z. Vlasov [Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 8, 109-140 (1944); these Rev. 7, 42] that the elastostatic problem of thin shallow shells can be reduced to the solution of two simultaneous

fourth-order partial differential equations for the stress and deflection functions. The author shows that Vlasov's system of two real equations is equivalent to a single fourth-order differential equation in the complex domain. This complex differential equation is reduced to an integral equation of the Volterra type in the complex domain, which can be solved by the method of successive approximations. The general solution of the integral equation can be represented by an analytic function of two complex variables. This is illustrated by two examples relating to sufficiently shallow spherical and cylindrical shells.

*I. S. Sokolnikoff* (Los Angeles, Calif.).

**Ambarcumyan, S. A. On the theory of anisotropic shallow shells.** Akad. Nauk SSSR. Prikl. Mat. Meh. 12, 75-80 (1948). (Russian)

The paper contains a derivation of the differential equations of equilibrium for thin shallow anisotropic shells whose materials have one plane of elastic symmetry parallel to the middle surface of the shell. The Kirchhoff-Love hypothesis is assumed in the derivation and it is supposed that the coefficients of the first quadratic differential form vary so slowly that they can be treated as constants in the differentiation. The considerations of this paper are analogous to S. G. Lehnitzky's formulation of the problem of deflection of thin anisotropic elastic plates [Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] (N.S.) 2, no. 2, 181-210 (1938)]. *I. S. Sokolnikoff* (Los Angeles, Calif.).

**Beskin, Leon. Warping and shear lag in closed cylindrical shells.** J. Aeronaut. Sci. 15, 221-231 (1948).

The state of stress in thin-walled closed cylindrical shells is determined in the neighborhood of sections with concentrated loads or of sections corresponding to discontinuities of section properties, such as fixed ends. A method of successive approximations is outlined using conventional beam formulas as first approximation. *E. Reissner.*

**Dei Poli, Sandro. Sulla stabilità elastica della striscia cilindrica compressa secondo le generatrici.** Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 81-82, 95-101 (1948).

**Bijlaard, P. P. On the torsional and flexural stability of thin walled open sections.** Nederl. Akad. Wetensch., Proc. 51, 314-321 (1948).

**Hemp, W. S. The theory of flat panels buckled in compression.** Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. 2178 (8764), 9 pp. (1945).

A method of approximate solution is proposed for the problem of finite deflections of an infinite plate strip with clamped or simply supported edges. The plate is assumed to be compressed in the direction of the edges of the strip, slightly in excess of the stress  $f_b$  which causes buckling. Calculations are carried out to the point where numerical values are obtained for the quantity  $(\partial f_a / \partial f_b)_{f_b=f_b}$ , where  $f_a$  is the average compressive stress and  $f_b$  the compressive stress at the edges of the plate. *E. Reissner.*

**Niordson, Frithiof I. N. Buckling of conical shells subjected to uniform external lateral pressure.** Acta Polytech., no. 14 = Trans. Roy. Inst. Tech. Stockholm 1947, no. 10, 23 pp. (1947).

A formula is obtained from the theory of thin shells to determine the buckling pressure of frustums of conical shells

for which a quantity  $\beta = l \cot \alpha / \rho$  is so small that powers of  $\beta$  higher than the second can be neglected. In the definition of  $\beta$ ,  $l$  is the slant height of the frustum,  $\alpha$  the inclination of a generator to the base and  $\rho$  the radius of curvature of the base. An energy method is used and several approximations, some based on observed behavior, are made to simplify the analysis. It is suggested that the introduction of a reduced modulus of elasticity will permit the application of the formula to buckling at stresses beyond the proportional limit.

*H. W. March* (Madison, Wis.).

**Stagni, Ernesto. Applicazione del metodo numerico alla ricerca delle frequenze di vibrazione libera di strutture lineari.** Ann. Mat. Pura Appl. (4) 26, 85-94 (1947).

**Mettler, E. Eine Theorie der Stabilität der elastischen Bewegung.** Ing.-Arch. 16, 135-146 (1947).

An isotropic homogeneous elastic medium is considered to be in a state of dynamic equilibrium and it is asked whether this state is stable under small perturbations (i.e., do such perturbations grow in time?). The problem is formulated by the variational approach treating the initial state as known so that the problem is linear in the perturbation quantities. The Euler equations are written down for a particular example.

*G. F. Carrier.*

**Pailloux, Henri. Sur la méthode de Rayleigh-Ritz pour les systèmes déformables.** C. R. Acad. Sci. Paris 226, 1882-1884 (1948).

This note deals with some generalities concerning the use of Lagrangian coordinates in the theory of small free vibrations of an elastic system about a configuration of equilibrium, and in the theory of forced vibrations of such systems. Emphasis is placed upon the relations between the problems concerning free vibrations and problems concerning extrema of quadratic forms in infinitely many variables. In the opinion of the reviewer most of the points made by the author are already rather familiar. *L. A. MacColl.*

**Sommerfeld, Arnold. Spezielle Lösungen des Problems der elastischen Eigenschwingungen beim Quader und Würfel.** S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. 1945/46, 81-88 (1947).

A family of eigenfunctions associated with the oscillations of a rectangular parallelepiped of isotropic elastic material are demonstrated. They are valid when one or more of the ratios of edge lengths are rational and correspond to the case where all faces are force free. The oscillations are characterized by the fact that the displacement is a two-component vector and that the dilatation [i.e., the divergence of the displacement] vanishes identically.

*G. F. Carrier* (Providence, R. I.).

**Bonfiglioli, Guido. Sulle vibrazioni libere di un portale elastico.** Pont. Acad. Sci. Comment. 6, 421-443 (1942).

**Vekua, I. N. On a method for the solution of the boundary problems of the sinusoidal oscillations of an elastic cylinder.** Doklady Akad. Nauk SSSR (N.S.) 60, 779-782 (1948). (Russian)

The method mentioned in the title consists of an application of results derived by the author in an earlier paper [Bull. Acad. Sci. Georgian SSR [Soobščenija Akad. Nauk Gruzinskoi SSR] 4, 207-214 (1943); these Rev. 6, 272].

*H. P. Thielman* (Ames, Iowa).

Rozovskil, M. I. The impact of a cylinder on the surface of a substance whose mechanical properties alter with time. *Doklady Akad. Nauk SSSR (N.S.)* 61, 25-28 (1948). (Russian)

Rozovskil, M. I. The application of integral and integro-differential equations to the study of the deformation processes of real materials. *Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk* 1948, 601-622 (1948). (Russian)

Using the Boltzmann-Volterra theory of elastic after-effect, the author treats longitudinal, transverse and torsional vibrations of prismatic or cylindrical bars.

W. Prager (Providence, R. I.).

Caloi, Pietro. Sulle onde di Rayleigh in un mezzo elastico, fermo-viscoso indefinito. *Pont. Acad. Sci. Acta* 10, 143-154 (1946).

Uflyand, Ya. S. The propagation of waves in the transverse vibrations of bars and plates. *Akad. Nauk SSSR. Prikl. Mat. Meh.* 12, 287-300 (1948). (Russian)

When the effects of shearing force and rotational inertia are taken into account, the equation (1)  $\partial^4 u / \partial x^4 + c \partial^4 u / \partial t^4 = 0$  for the lateral vibrations of a prismatic bar becomes

$$(2) \quad \frac{\partial^4 u}{\partial x^4} - \left( \frac{1}{c_1^2} + \frac{1}{c_2^2} \right) \frac{\partial^4 u}{\partial x^2 \partial t^2} + \frac{1}{c_1^2 c_2^2} \frac{\partial^4 u}{\partial t^4} + c \frac{\partial^2 u}{\partial t^2} = 0.$$

This equation is solved by using a Laplace transformation. The solution consists of two waves propagated along the bar with velocities  $c_1$  and  $c_2$ . The author considers a semi-infinite bar, suddenly loaded at the end, and a finite hinged bar suddenly loaded at the midpoint, with a concentrated constant force. For large values of the time  $t$ , the solutions of (2) converge to the corresponding equilibrium solutions of (1). The transient state, however, can be found only

from (2). The same effects are considered for the lateral vibrations of a plate and analogous results are found.

W. H. Muller (Amsterdam).

\*Bullen, K. E. *An Introduction to the Theory of Seismology*. Cambridge, at the University Press; New York, The Macmillan Company, 1947. xiv+276 pp. \$4.00.

This excellent book compresses into a brief space a surprising range of material, both theoretical and practical. Each chapter is a summary treatise, a mathematical essay, combining in a graceful way existing theory in a uniform dress with the author's own contributions and comments.

The chapter on elasticity theory not only presents in the terse language of tensors the theory of stress and strain in an isotropic perfectly elastic solid and the modifications that must be introduced to allow for departures from those specifications, but points out the parallelisms and analogies in the treatment of plastics and fluids. There is a chapter on the general theory of vibrations and of plane and spherical waves, followed by chapters on bodily elastic waves in the earth, and surface elastic waves, on the reflection and refraction of elastic waves, on seismic rays in a spherically stratified earth model and on amplitudes of surface motion due to seismic waves in a spherically stratified earth model.

The author devotes one chapter to the principle of the seismograph, restricting the discussion to first order effects. He has chapters on the construction of travel-time tables in which he has taken so active a part, on the seismological observatory, on seismology and the earth's upper layers, on seismology and the earth's deep interior, and on earthquake occurrence. In the last chapter the author throws together brief discussions of such diverse topics as earthquake field studies, seismic sea waves, seiches, seismological engineering, microseisms, seismic prospecting and atom bombs.

J. B. Macelwane (St. Louis, Mo.).

## MATHEMATICAL PHYSICS

### Optics, Electromagnetic Theory

Banerjee, Durga Prosad. On the harmonics associated with an ellipsoid and its application to the electrification of two parallel coaxial elliptic discs. *J. Math. Pures Appl.* (9) 26 (1947), 269-282 (1948).

The author first lists some formulas for the ellipsoidal harmonic functions introduced by S. K. Banerjee [Bull. Calcutta Math. Soc. 10, 95-104, 179-186 (1919)], and then discusses the electrification of two parallel coaxial elliptic discs in terms of these functions. His method is exactly parallel to that used by J. W. Nicholson [Philos. Trans. Roy. Soc. London. Ser. A 224, 303-369 (1924)] for circular discs, and the resulting formulas for the potential and surface charge reduce to Nicholson's in the limiting case of zero eccentricity.

M. C. Gray (New York, N. Y.).

Valatin, Jean G. Les formes bilinéaires du champ de Maxwell. *C. R. Acad. Sci. Paris* 227, 39-41 (1948).

It is shown that Maxwell's equations can be written in the form

$$i \frac{\partial}{\partial t} \Phi = \frac{c}{i} \left( \frac{\partial}{\partial x} \sigma_x \epsilon + \frac{\partial}{\partial y} \sigma_y \epsilon + \frac{\partial}{\partial z} \sigma_z \epsilon \right) \Phi,$$

where  $\Phi = (i\mathbf{E}, \mathbf{H})$  and the  $\sigma$ 's and  $\epsilon$  are the matrices

$$\sigma_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix},$$

$$\sigma_z = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \epsilon = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Functions bilinear in  $\Phi$  and  $\Phi^*$  are listed and their transformation properties discussed.

C. Kikuchi.

Valatin, Jean G. La polarisation circulaire et l'opérateur rotationnel du champ de Maxwell. *C. R. Acad. Sci. Paris* 227, 110-112 (1948).

Valatin, Jean G. Le double aspect des équations de Maxwell dans la théorie quantique du rayonnement. *C. R. Acad. Sci. Paris* 227, 177-179 (1948).

The properties of the equation [see the preceding review]  $i(\partial/\partial t)\Phi = c \operatorname{rot} \epsilon \Phi$  are developed. In particular, the consequences of taking the field Hamiltonian to be the linear operator  $H = c \hbar \operatorname{rot} \epsilon$  are considered. Some of the results are: the eigenvalues of the velocity operator are  $\pm c$ , and 0; the operators  $L_x = -i\hbar(y\partial/\partial z - z\partial/\partial y)$ , etc., do not commute with the Hamiltonian, although  $L_x + \hbar\sigma_x$ ,  $L_y + \hbar\sigma_y$ , and  $L_z + \hbar\sigma_z$  do commute. C. Kikuchi (East Lansing, Mich.).

Giorgi, Giovanni. *Struttura intrinseca del campo elettromagnetico*. Pont. Acad. Sci. Acta 9, 95-102 (1945).

Flamm, Ludwig. *Der Mechanismus elektromagnetischer Wellen. I. Ebene Wellen*. Akad. Wiss. Wien, S.-B. IIa. 154, 1-17 (1945).

A theory of the electromagnetic field based on the mechanical properties of the electric and magnetic lines of force is developed. [See also *Österreich. Ing.-Arch.* 1, 105-117 (1946); these Rev. 8, 181.] It is shown that for a plane wave the *D*-vector lies on the surface

$$(1) \quad Q(x, y, t) = D_0 y + \partial \pi(x, t) / \partial x = \text{constant},$$

where  $\pi(x, t)$  is an arbitrary solution of

$$(2) \quad \partial^2 \pi / \partial x^2 = c^{-2} \partial^2 \pi / \partial t^2.$$

It is then shown that  $y$  also satisfies the wave equation (2) and that  $H = -k D_0 \partial y / \partial t$ . From these results, the interpretation is given that the electric lines of force are like vibrating strings and the magnetic field like transverse velocity. It is also pointed out that the *D*-line has an equivalent mass density of  $\rho_m = 2w_1/c^2$ , where  $w_1$  is the electric energy density. *C. Kikuchi* (East Lansing, Mich.).

Flamm, Ludwig. *Der Mechanismus elektromagnetischer Wellen. II. Kugelwellen*. Akad. Wiss. Wien, S.-B. IIa. 154, 18-49 (1945).

The theory developed in the paper reviewed above is extended to spherical electromagnetic waves.

*C. Kikuchi* (East Lansing, Mich.).

Flamm, Ludwig. *Die Linienmechanik der elektrischen Feldmaterie*. Akad. Wiss. Wien, S.-B. IIa. 155, 221-279 (1947).

The theory developed in the author's earlier papers [see the two preceding reviews] is summarized and amplified, in particular, by considering an axially symmetric electromagnetic field. The solution of the differential equation  $d^2R/dr^2 + (k^2 - \lambda/r^2)R = 0$  which arises in this case is discussed in detail. *C. Kikuchi* (East Lansing, Mich.).

Fok, V. A. *The propagation of the direct wave around the earth taking account of diffraction and refraction*. Izvestiya Akad. Nauk SSSR. Ser. Fiz. 12, 81-97 (1948). (Russian)

The propagation of radio waves emitted by a dipole located on the surface of the earth depends on three factors: diffraction around the earth, refraction through the atmosphere, reflection by the ionosphere. It is interesting to have a solution of Maxwell's equations for the part due to the first two factors (direct wave). If we denote by  $h$  the height of the point under consideration and by  $s$  its distance from the source, projected on the earth's surface, we can introduce the dimensionless coordinates  $y = h/h_1$ ,  $x = s/s_1$ , with  $h_1 = (\frac{1}{2}a^*/k^2)$ ,  $s_1 = (2a^*/k)$ ,  $k$  being the value of  $2\pi/\lambda$  at the surface of the earth and  $a^*$  the so-called equivalent radius of the earth. Introducing now the function  $W_1$ , related to Hertz's function  $U$  by  $W_1 = e^{-i\omega t}(\epsilon/\epsilon_0)(s_1/a)^{1/2}(\sin s/a)^{1/2}U$ , where  $\epsilon$  is the dielectric constant of the atmosphere,  $\epsilon_0$  its value near the ground and  $a$  the radius of the earth, one finds the approximate equation  $\partial^2 W_1 / \partial y^2 + i\omega \partial W_1 / \partial x + y(1+g)W_1 = 0$ . Here  $g$  denotes the dimensionless quantity

$$(\frac{1}{2}a^*/\epsilon_0)((\epsilon - \epsilon_0)h^{-1} - \epsilon_0'),$$

$\epsilon'$  being the gradient of  $\epsilon$  at the surface of the earth. If  $\eta$  denotes the complex dielectric constant of the ground and

$q = ikh_1(\epsilon_0/\eta)^{1/2}$ , the boundary condition at the surface of the earth is  $\partial W_1 / \partial y + qW_1 = 0$  ( $y = 0$ ). The solution satisfying this condition and having the required singularity in the neighborhood of the source is

$$W_1 = e^{i\omega t/L} \int_{\Gamma} e^{i\omega t} \frac{f(y, t)}{(\partial f / \partial y + qf)_{y=0}} dt.$$

Here  $f(y, t)$  is that solution of the differential equation  $d^2f / dy^2 + (y - t + yg)f = 0$  which, for large values of  $y - t$ , admits the asymptotic representation

$$f(y, t) = Ce^{i\omega t}(y - t + yg)^{-1} \exp \left[ i \int_{\Gamma} (u - t + uy) du \right]$$

and  $\Gamma$  is a contour coming from and going to infinity and leaving all the zeros of the denominator on the same side. Discussion shows that to a close approximation the direct wave is propagated above the horizon according to the laws of geometrical optics, while diffraction takes place only below the horizon. *G. Toraldo di Francia* (Florence).

Malyuzhinec, G. D. *On a generalization of the formula of Weyl for a wave field above an absorbing plane*. Doklady Akad. Nauk SSSR (N.S.) 60, 367-370 (1948). (Russian)

Let  $u = u_1 + u_2$  represent a scalar wave field in the half-space  $z > 0$ , above the absorbing plane  $z = 0$ . The incident part  $u_1$  is known, being produced by a given distribution of sources in the half-space  $z > 0$ . The problem is to find the reflected part  $u_2$ , with the boundary condition  $\partial u / \partial z + ikgu = 0$  for  $z = 0$ . Following the analogy with the case  $|g| = \infty$ , one can establish the formula

$$u_2(x, y, z) = u_1(x, y, -z) - 2ikge^{-ikz} \int_{i\infty}^0 e^{ikz} u_1(x, y, -z) dz.$$

Weyl's formula for an incident spherical wave is shown to be a particular case. *G. Toraldo di Francia* (Florence).

Heins, Albert E. *The radiation and transmission properties of a pair of semi-infinite parallel plates*. I. Quart. Appl. Math. 6, 157-166 (1948).

This paper is a continuation of similar work by the author in collaboration with J. F. Carlson [same Quart. 4, 313-329 (1947); 5, 82-88 (1947); these Rev. 8, 422, 614]. The author is now concerned with the following problem. A plane monochromatic electromagnetic wave is incident upon a pair of nonstaggered parallel half-planes both of perfectly conducting material and zero thickness. The direction of propagation of the incident wave is perpendicular to the edges of the plates while the electric vector is parallel to them. Thus the diffraction problem is essentially scalar and two-dimensional. It is further assumed that  $\frac{1}{2}a/\lambda < 1$  ( $a$ , distance between the plates;  $\lambda$ , wave length of incident wave) so as to guarantee that only one travelling wave is generated between the plates. The author's main purpose is to calculate the transmission coefficient, that is, the amplitude and phase of the travelling wave far away from the edges. This is accomplished after first solving a pair of simultaneous integral equations for the two surface-current densities on the plates. These integral equations are of the Faltung type, closely related to those of Wiener and Hopf. The technique of solution is similar to that applied in the papers cited. Finally, by virtue of the Lorentz reciprocity theorem, the magnitude of the transmission coefficient immediately gives the radiation pattern in the reverse problem, that is, when the slit acts as an antenna fed by an incident wave travelling between the plates towards the edges.

*C. J. Bouwkamp* (Eindhoven).

Opechowski, W. *Electromagnetic waves in wave guides. I. General theoretical principles; rectangular wave guides.* Philips Tech. Rev. 10, 13-25 (1948).

The author remarks that transmission line theory in the microwave region must be based upon the Maxwell equations. He first surveys the properties of plane waves (electromagnetic) in an unbounded conducting medium and then discusses the theory of rectangular wave guides.

A. E. Heins (Pittsburgh, Pa.).

Opechowski, W. *Electromagnetic waves in wave guides. II. Coaxial cables and circular wave guides.* Philips Tech. Rev. 10, 46-54 (1948).

The author discusses the transmission line aspects of circularly symmetric structures, that is, the circular wave guide, the coaxial cable and the round wire, by starting directly from the Maxwell equations. A. E. Heins.

Rice, S. O. *Reflections from circular bends in rectangular wave guides. Matrix theory.* Bell System Tech. J. 27, 305-349 (1948).

The author presents a new method of computing the reflection produced by a circular bend in a rectangular wave guide. The electromagnetic field is derived from two scalar wave functions representing the components of the vector potentials normal to the plane of the bend. In the curved part of the guide suitable coordinates are introduced which closely resemble the ordinary rectangular coordinates in the straight parts of the guide; thus the use of Bessel functions [cf. H. Buchholz, Elektr. Nachr. Techn. 16, 73-85 (1939)] is avoided.

The coefficients of the different modes occurring in the vector potentials are arranged as infinite column matrices. An infinite square matrix is constructed that plays the same role in the multiple-mode propagation in the wave guide as does the propagation constant in a simple transmission line. The continuity conditions at the junctions of the bent and the straight parts lead to the matrix equivalent of an infinite set of linear equations for the coefficients. These matrix equations are simple from a formal point of view; however, numerical solution is cumbersome though not impossible as the author illustrates by an example. By a limiting process, approximate formulas pertaining to gentle bends are obtained from the general matrix equations. The propagation and reflection constants for the dominant modes are given when either the electric or the magnetic vector is in the plane of the bend. The reflection formula for the *E*-bend is new, while that for the *H*-bend generalizes an unpublished result of R. E. Marshak.

Tables show numerical results for reflection and transmission coefficients for various curvatures of the bend; auxiliary functions are tabulated in an appendix.

C. J. Bouwkamp (Eindhoven).

Abele, Manlio. *Integrazione approssimata delle equazioni de Maxwell nell'interno di una cavità risonante.* Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 81-82, 159-167 (1948).

De Simoni, Franco. *Teoria matematica dei risonatori cavi cilindrici eccitati da un dipolo hertziano.* Pont. Acad. Sci. Comment. 9, 491-513 (1945).

De Simoni, Franco. *Teoria matematica dei risonatori cavi prismatici eccitati da un dipolo hertziano.* Pont. Acad. Sci. Comment. 10, 249-269 (1946).

Graffi, Dario. *Sulla propagazione delle onde elettromagnetiche entro tubi conduttori.* Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) 2, 47-53 (1946).

Kahan, Théo, et Colombo, Serge. *Étude des régimes transitoires dans les guides d'ondes et les cavités électromagnétiques.* C. R. Acad. Sci. Paris 226, 2060-2061 (1948).

Paterno, Cariotta. *Formule risolutive per i problemi generali sulle reti di conduttori elettrici.* Pont. Acad. Sci. Acta 6, 25-33 (1942).

Gavrilov, M. A. *Transformation of relay-contact schemes of class H.* Doklady Akad. Nauk SSSR (N.S.) 59, 1579-1582 (1948). (Russian)

The author regards a complicated relay-contact scheme as a collection of simpler relay networks  $X_1, \dots, X_n$  which have been interconnected. When the interconnections are of certain particularly simple types (analogous to series or parallel connections) the algebra of logic may be used to describe the scheme directly in terms of the  $X_i$  and to transform it to equivalent circuits built up of the  $X_i$  without first transforming to an equivalent series-parallel relay scheme. E. N. Gilbert (Murray Hill, N. J.).

Kuznecov, P. I. *The propagation of electromagnetic waves along two parallel single-conductor lines.* Akad. Nauk SSSR. Prikl. Mat. Meh. 12, 141-148 (1948). (Russian)

The propagation of electromagnetic waves in a multi-conductor system depends on the equations

$$-\frac{\partial}{\partial x}(V) = [R](J) + [L]\frac{\partial}{\partial t}(J),$$

$$-\frac{\partial}{\partial x}(J) = [G](V) + [C]\frac{\partial}{\partial t}(V),$$

where  $(V)$  and  $(J)$  are one-column matrices, whose  $n$  elements correspond respectively to the potentials and currents of the  $n$  conductors, while  $[R]$ ,  $[L]$ ,  $[G]$  and  $[C]$  are square matrices of order  $n$ , corresponding to resistance, inductance, conductance and capacity. In the case  $n=2$ , with the initial condition of zero potential and current all along the conductors, the author shows that a formal solution is possible by means of a Laplace transformation. In particular, in the case of two parallel semi-infinite conductors the free ends of which are maintained respectively at potentials 1 and 0, the solution can be given explicitly in the form of contour integrals. The author then shows that this formal solution has real meaning and satisfies all the conditions of the problem. As a numerical example tables and graphs are given for the symmetrical case.

G. Toraldo di Francia (Florence).

### Quantum Mechanics

Drăganu, Mircea. *Remarque sur l'équation de Schrödinger en coordonnées quelconques.* C. R. Acad. Sci. Paris 226, 1802-1803 (1948).

Ciriquian, Jose Estevan. *Various methods for determining wave functions in atomic systems.* Revista Acad. Ci. Zaragoza (2) 25-26 (1947). (Spanish)  
Expository article.

Pirenne, Jean. *La méthode des perturbations en théorie des champs quantifiés et la construction de la matrice  $S$  de Heisenberg.* Helvetica Phys. Acta 21, 226-232 (1948).

Borghi, D. C. *Sui principi della fisica nucleare*. Pont. Acad. Sci. Comment. 10, 145-215 (1946).

de Wet, J. S. *On the quantization of field theories derived from higher order Lagrangians*. Proc. Cambridge Philos. Soc. 44, 546-559 (1948).

The theory of quantization of fields is extended to the case of Lagrangians in which higher order derivatives than the first appear. It is shown how to define canonical variables so that the equations may be put in Hamiltonian form. The theory is quantized as usual by regarding these variables as operators and postulating commutation relations between them. It is shown that these relations are Lorentz invariant, consistent with the equations of motion, and independent of the choice of the family of parallel space-like hyperplanes with which the canonical variables were associated. These results are established for both Bose-Einstein and Fermi-Dirac quantization. *H. C. Corben*.

Flint, H. T., and Symonds, N. *The conservation of energy, momentum and charge in the nuclear field*. Philos. Mag. (7) 39, 413-419 (1948).

Flint's theory of the nuclear field [same Mag. (7) 38, 22-32 (1947); these Rev. 9, 167] is slightly modified and the conservation laws of momentum, energy and charge developed. In a five-dimensional theory such as this, these laws assume a unified form, the components  $\nu=5$ ,  $\mu\neq 5$  of the total energy tensor  $\Pi$ ,  $\nu$  being proportional to the current density. It then follows that this four-dimensional vector is conserved and that the other components  $\Pi_{\mu}^{\nu}$  satisfy the correct conservation laws even in the presence of an electromagnetic field, if  $\Pi_{\mu}^{\nu}$  is regarded as a superposition of the electromagnetic and nuclear energy tensors of the particle. It is shown that the total energy of particle and field  $-\int \Pi^{\mu} d\nu$  may be regarded as the energy of the nucleon together with additional energy acquired by the nucleon because of its interaction with the field. This interpretation leads to the result that the nucleon in a scalar field such as that considered here would appear to change its mass.

*H. C. Corben* (Pittsburgh, Pa.).

Balseiro, José A. *Transformation theory applied to a radiation field*. Physical Rev. (2) 73, 1346-1348 (1948).

Having given a photon distribution described by a field function in terms of an orthogonal system, the question arises as to what will be the photon distribution when the field is given in terms of another orthogonal system of functions. The problem is solved by means of transformation theory.

*Author's summary.*

Williamson, Marjorie. *A note on M. de Broglie's theory of the photon*. Philos. Mag. (7) 39, 314-324 (1948).

The author describes a photon by a two-index four-component spinor, that is, by a wave function with 16 components. This wave function is assumed to satisfy a first-order differential equation similar to Dirac's equation for the electron in which a mass term appears. The wave function may be represented as a  $4\times 4$  matrix and this in turn may be written as a linear combination of the Dirac matrices  $\alpha_i$  ( $i=1, 2, 3, 4$ ) and their products. Although the author does not expand the wave function in this manner explicitly she does identify the coefficients of  $\alpha_i$  in this expansion with the vector potential and those of  $\alpha_i \alpha_j$  with the electromagnetic field quantities describing the photon. The remaining six components are not interpreted physically. A Hamiltonian operator is proposed and the spin of the photon is discussed. *A. H. Taub* (Urbana, Ill.).

Serpé, J. *Sur le problème de la self-énergie de l'électron dans la théorie de Gustafson*. Physica 14, 223-236 (1948).

Following Gustafson's use of M. Riesz's method of analytic continuation [Kungl. Fysiografiska Sällskapets i Lund Forhandlingar [Proc. Roy. Physiog. Soc. Lund] 15, no. 28, 277-288 (1945); these Rev. 7, 180] in attempting to obtain a convergent approximation to the electromagnetic self-energy of a point electron, the author derives results for this problem agreeing with those of Nilsson [Physical Rev. (2) 73, 903-909 (1948); these Rev. 9, 557]. It is shown that Gustafson's neglect of the effects of retardation is wrong, particularly where the intermediate states for the electron have negative energy. Adequate symmetrization is necessary to avoid obtaining imaginary mean values for Hermitian quantities. The theory is that of a single electron, not "hole" theory. The method does not eliminate certain oscillating semi-convergent integrals. For an electron at rest the self-energy is zero to order  $e^2$ .

*C. Strachan* (Aberdeen).

Sengupta, N. D. *On an exact solution of Dirac electron in the field of electromagnetic radiation*. Bull. Calcutta Math. Soc. 39, 147-153 (1947).

The author obtains the exact solution of the Dirac equation for an electron in the field of a plane electromagnetic wave. He uses the invariant formulation of the Dirac equation and obtains the solution more readily than does Volkov who solved the problem earlier [Z. Physik 94, 250-260 (1935)]. The four-dimensional current vector is computed for the solution obtained. *A. H. Taub* (Urbana, Ill.).

Rubinowicz, A. *Dirac's one-electron problem in momentum representation*. Physical Rev. (2) 73, 1330-1333 (1948).

Majumdar, R. C., and Gupta, S. N. *On the self-energy of the electrons*. Proc. Nat. Inst. Sci. India 13, 187-195 (1947).

Petiau, Gérard. *Sur l'équation d'ondes non relativiste des corpuscules de spin  $h/4\pi$  dans un champ nucléaire général*. C. R. Acad. Sci. Paris 227, 263-264 (1948).

Petiau, Gérard. *Sur le passage des ondes corpusculaires de spin  $(1/2) \cdot (h/2\pi)$  à travers les barrières de champs nucléaires*. Revue Sci. 85, 1094-1106 (1947).

### Thermodynamics, Statistical Mechanics

de Broglie, Louis. *Sur la variance relativiste de la température*. Cahiers de Physique nos. 31-32, 1-11 (1948).

Let an amount  $Q$  of heat be added to a body moving with uniform velocity  $v=\beta c$ , and let  $Q_0$  be the amount of heat that must be added to this same stationary body to produce the same change in its proper mass. It is shown that  $Q=Q_0(1-\beta^2)^{1/2}$ , and hence that

$$(*) \quad E = E_0(1-\beta^2)^{-1/2}, \quad T = T_0(1-\beta^2)^{1/2},$$

i.e., that energy and temperature transform respectively in the same manner as undulatory frequency  $\nu$  and cyclic frequency  $\nu_0$ . For a periodic system of frequency  $\nu_0$ , temperature and entropy may be defined consistently in terms of action by the relations  $kT_0 = \nu_0 A_0$ ,  $S_0 = k \log A_0$ . For a system moving with velocity  $\beta c$ ,  $kT = \nu_0 A_0$ ,  $S = k \log A_0$ . If  $A_0$  is

assumed to be an integer multiple of  $\hbar$ , it follows from (\*) that  $E = nh\nu$ ,  $kT = nh\nu$ . These relations provide a basis for a wave-mechanical theory of thermodynamics.

C. C. Torrance (Annapolis, Md.).

**Wang, J. S.** Thermodynamics of equilibrium and stability. Chinese J. Phys. 7, 132-175 (1948).

This paper seeks to improve the classical treatment by a recombination of standard techniques. C. C. Torrance.

**Wang, J. S.** Free energy in the statistical theory of order-disorder transformation. Sci. Rep. Nat. Tsing Hua Univ. 4, 341-360 (1947).

**Bopp, Fritz.** Quantenmechanische Statistik und Korrelationsrechnung. Z. Naturforschung 2a, 202-216 (1947).

The author proves that the quantum mechanical uncertainty principle precludes the use of bivariate distribution functions for canonically conjugate quantities. This is connected with the fact that the measurement of one quantity changes the distribution of the other. Nevertheless it is of interest to compute the correlations of such quantities. This is achieved by a generalization of Gebelein's method [Z. Angew. Math. Mech. 21, 364-379 (1941); these Rev. 4, 104]. Let two quantities  $X$ ,  $Y$  assume the values  $x_\mu$  and  $y_\nu$  ( $\mu, \nu = 0, 1, \dots, N-1$ ); let  $W_{\mu\nu}$  be a bivariate probability function;  $u_\mu = \sum_\nu W_{\mu\nu}$ ,  $v_\nu = \sum_\mu W_{\mu\nu}$  and  $\sum u_\mu = \sum v_\nu = 1$ . Consider the transformations  $\xi_\mu = \xi(x_\mu)$ ,  $\eta_\nu = \eta(y_\nu)$ . The conventional correlation coefficient  $r^2 = \sum_{\mu, \nu} \xi_\mu \eta_\nu W_{\mu\nu} / \sum_\mu \xi_\mu^2 \sum_\nu \eta_\nu^2 \leq 1$  depends on the above transformation. Gebelein's correlation coefficient is  $K = \max r$ . This variational problem leads to an eigenvalue problem, as follows. A correlation matrix  $S$  is defined:  $W_{\mu\nu} = \psi_\mu S_{\mu\nu} \chi_\nu$ , where  $\psi_\mu = u_\mu^{1/2}$ ,  $\chi_\nu = v_\nu^{1/2}$  are probability amplitudes appearing already in classical statistics. Furthermore,  $f_\mu = \psi_\mu \xi_\mu$ ,  $g_\nu = \chi_\nu \eta_\nu$ , and  $Sg = \rho f$ ,  $S'f = \rho g$  ( $S'$  is the transposed of  $S$ ). Let  $\rho^2$  be the eigenvalue of  $SS'$ . The largest eigenvalue is  $\rho^2 = 1$ , the second largest  $\rho^2 = K^2$ . The author generalizes this method by admitting any complex  $S$  matrix. One has  $S^* \psi = \rho \chi$ ,  $S \chi = \rho \psi$  ( $S^*$  is the Hermitian adjoint of  $S$ ) and  $SS^* \psi = \rho^2 \psi$ . One has

$$W_{\mu\nu} = (2\rho)^{-1} [\psi_\mu^* S_{\mu\nu} \chi_\nu + \psi_\nu S_{\mu\nu}^* \chi_\mu^*]$$

(\* means conjugate complex). If  $W_{\mu\nu} > 0$  for all  $\mu, \nu$ , one arrives at Gebelein's results. If  $S$  is unitary, one obtains the quantum mechanical correlation between canonical variables. More generally one gets a Gibbs ensemble of quantum mechanical systems. The formalism also contains cases for which no physical interpretation has been found.

L. Tisza (Cambridge, Mass.).

**Schubert, Gerhard.** Zur Bose-Statistik (Nachtrag). Z. Naturforschung 2a, 250-251 (1947).

Continuation of same Z. 1, 113-120 (1946); these Rev. 8, 556.

**Gurov, K. P.** On quantum hydrodynamics. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. 18, 110-125 (1948). (Russian)

This paper is concerned with the derivation of the macroscopic hydrodynamical equations of a normal liquid from a quantum-theoretical model. The author hopes eventually to justify, but does not use, the phenomenological "quantum hydrodynamics" of Landau [Acad. Sci. USSR. J. Phys. 5, 71-90 (1941)]. The method of the paper is said to be a combination of the statistical-mechanical treatment of a classical liquid by N. Bogolyubov and a formulation of statistical quantum theory by the author; neither of these papers was accessible to the reviewer.

The treatment appears to be on orthodox lines, using the von Neumann density matrix to pass from the microscopic to the macroscopic picture. Nonequilibrium configurations of the liquid are represented by expanding all local distribution functions in powers of a parameter  $\mu$ , which is small and slowly varying in space and time; the terms in  $\mu^0$  represent local equilibrium conditions, terms in  $\mu^1$  small deviations due to transport phenomena, and so on. Detailed analysis is given only for monatomic spherically symmetrical molecules, neglecting many-body interactions. The zero-order approximation gives then the classical equations of motion of a perfect fluid. The first-order approximation should give the laws of viscosity and heat-conduction, with coefficients calculable from the molecular interactions, but its consideration is deferred to a future paper.

F. J. Dyson (Princeton, N. J.).

**Perelesin, A. S.** On the theory of the crystalline state. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. 18, 449-456 (1948). (Russian)

Vlasov's equation [Acad. Sci. USSR. J. Phys. 9, 130-138 (1945); these Rev. 7, 183] for the distribution function of an assembly of identical uncharged particles subject to central force interactions reduces, in the case of a stationary state of a system whose distribution function depends on the velocities only through the kinetic energy, to the integral equation:

$$\rho(\mathbf{r}) = C(T) \exp \left\{ -\frac{1}{kT} \int_{-\infty}^{\infty} K(|\mathbf{r} - \mathbf{r}'|) \rho(\mathbf{r}') d\mathbf{r}' \right\}.$$

Here  $\rho$  is the spatial density distribution of the particles,  $K(|\mathbf{r} - \mathbf{r}'|)$  is their energy of interaction,  $k$  is Boltzmann's constant,  $T$  the absolute temperature and  $C(T)$  a normalizing factor dependent on  $T$ . Vlasov has shown [loc. cit.] that as  $T \rightarrow 0$  in an infinite cubic lattice the above equation has only periodic solutions with the period of the lattice. An increase of  $T$  leads to the broadening of the maxima of these periodic solutions. Vlasov has also said to have shown [Vestnik Moskov. Univ. 1946, no. 3-4, 63 ff.] that at higher temperatures in addition to this broadening new solutions appear which do not have the periodicity of the lattice. The present paper shows that these new aperiodic solutions appear in a discontinuous manner as the temperature rises above a certain minimum value estimated to be about a tenth of the temperature of fusion.

G. M. Volkoff.

**Ugarov, V. A.** On the theory of strata. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. 18, 457-461 (1948). (Russian)

The properties of an assembly of interacting particles are investigated with the aid of a modified Maxwell-Boltzmann equation. A current of mutually interacting particles (predominantly repulsive forces) with initially uniform spatial distribution becomes unstable and goes over into one with a periodically varying spatial distribution as soon as the drift velocity exceeds a certain critical value. This result is used to explain experimentally known periodic structures of beams (experiments of Merrill and Webb, strata in a discharge tube). Generalizations of this result are given for mixtures of different kinds of particles, for electrons in a metal, and for neutral particles. The present method differs from the usual Maxwell-Boltzmann procedure: (a) by neglect of the shock term in the equation, and (b) by inclusion of the interaction between particles by Vlasov's method.

G. M. Volkoff (Vancouver, B. C.).

the  
pic  
ons  
ou-  
small  
re-  
ialed  
cal  
ro-  
of  
ion  
ith  
out

ite.  
18,

30-  
ion  
to  
ary  
on  
ral

lea,  
an's  
mal-  
it.]  
ion  
ice.  
a of  
own  
her  
ons  
ice.  
ons  
ises  
t a

auk  
18).

are  
ann  
pre-  
tial  
with  
the  
sult  
ures  
n a  
for  
n a  
fers  
by  
by  
ov's